## *6.3 Proving Trigonometric Identities*

*Verifying versus Proving an Identity*

Verifying that an identity is true can be done either numerically (substituting in a value) or graphically. This gives some evidence that the statement suggests an identity but does not offer a "proof". The most complete method for proving trigonometric identities uses algebraic and trigonometric identities that have previously been established. This method can involve simplifying, factoring, and re-writing expressions.

Example 1: a) Verify that $\frac{\sin\theta + \tan\theta}{\cos\theta + 1} = \tan\theta$ is an identity

*i*) by substituting $\theta = \frac{\pi}{6}$

*ii*) by graphing for $-2\pi \le \theta \le 2\pi$

b) Use the basic identities to prove $\frac{\sin\theta + \tan\theta}{\cos\theta + 1} = \tan\theta$ for all permissible values of $\theta$. State any restrictions on $\theta$

$\cos\theta \neq -1$ $\quad \theta \neq \pi + n2\pi$

$\cos\theta \neq 0$ $\quad \theta \neq \frac{\pi}{2} + n\pi$

| $\frac{\sin\theta + \tan\theta}{\cos\theta + 1}$ | $\tan\theta$ |
|---|---|
| $\frac{\cos\theta\left(\sin\theta + \frac{\sin\theta}{\cos\theta}\right)}{\cos\theta\left(\cos\theta + 1\right)}$ <br> $\frac{\sin\theta\cos\theta + \sin\theta}{\cos\theta(\cos\theta+1)}$ <br> $\frac{\sin\theta(\cancel{\cos\theta+1})}{\cos\theta(\cancel{\cos\theta+1})}$ <br> $\frac{\sin\theta}{\cos\theta}$ | $\frac{\sin\theta}{\cos\theta}$ <br> RHS = LHS ✓ |

$a \cdot b + a$

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$a(b+1)$

Several strategies that are often successful were used in this proof.

a) Rewriting everything in terms of sine and cosine
b) Multiplying by one (using a common common denominator)
c) Only expanding when necessary
d) Try and make the more complicated side look like the less complicated side
e) Performing allowable operations with fractions

There is often more than one correct way to prove an identity. Sometimes it helps to work on **both** sides of the equation until they simplify to the same expression.

Example 2: Prove $\dfrac{\csc\theta - 1}{\cot\theta} = \dfrac{\cot\theta}{\csc\theta + 1} = \dfrac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta} + 1}$ State any restrictions on $\theta$

$\cot \neq 0$ $\quad \csc\theta \neq -1$

$\sin\theta \neq 0$

**Note:**

- In this example the two sides appear symmetrical: there is no "harder" side to start on!
- Expressions such as $(\sin\theta - 1)$ and $(\sin\theta + 1)$ are called the **conjugates** of each other. Multiplying them sometimes produces a Pythagorean Identity: $(\sin\theta - 1)(\sin\theta + 1)$

$$= \sin^2\theta - 1$$
$$= \cos^2\theta$$

$= (a+b)(a-b)$

$= a^2 - b^2$

| $\dfrac{\csc\theta - 1}{\cot\theta}$ | $\dfrac{\cot\theta}{\csc\theta + 1}$ |
|---|---|
| | $\dfrac{\cot\theta}{\csc\theta + 1} \times \dfrac{\csc\theta - 1}{\csc\theta - 1}$ |
| | $\dfrac{\frac{\cos\theta}{\sin\theta}\left(\frac{1}{\sin\theta} - 1\right)}{\csc^2\theta - 1}$ |
| * $\sin^2\theta + \cos^2\theta = 1$ <br> $1 + \cot^2\theta = \csc^2\theta$ | $\dfrac{\cancel{\cot\theta}(\csc\theta - 1)}{\cot^2\theta}$ |
| | $\dfrac{\csc\theta - 1}{\cot\theta}$ ✓ |
| | RHS = LHS. |

Further suggestions

- The use of the conjugate in some proofs is based on the principle of multiplying by 1. Do this to one side of the identity only.
- Once an identity is established, it can be rearranged. For example $\cot^2\theta = \csc^2\theta - 1$ is just another version of $\cot^2\theta + 1 = \csc^2\theta$, and any rearrangement can be used in future proofs.
- Do not combine more than one step in a proof on the same line. Your reasoning will **not** be clear and you may be penalized.
- If second degree terms are involved (ex. $\sin^2 x$), consider using the Pythagorean Identities or factoring . **Do not** take the square root of one side.

- Reciprocal and Quotient Identities can be generalized; for example:

  $\csc^2 x = \dfrac{1}{\sin^2 x}$ $\quad 5\csc 2x = \dfrac{5}{\sin 2x}$ $\quad 3\cot 2x = \dfrac{3\cos 2x}{\sin 2x}$ *or* $3\cot 2x = \dfrac{3}{\tan 2x}$

- Avoid common mistakes, such as

  $\dfrac{\cos 2x}{2} \neq \cos x$,

  $\sin^2 x + \cos^2 x = 1,\ \sin x + \cos x \neq 1$,

  $\sin 2x \neq 2\sin x \neq \sin^2 x$.

Example 3: Prove the identities and state any restrictions:

a) $\dfrac{\csc x - \sin x}{\cot x} = \cos x$ $\qquad$ b) $\dfrac{\sec x}{\tan x + \cot x} = \sin x$

$$\frac{\sin x\left(\frac{1}{\sin x} - \sin x\right)}{\sin x\left(\frac{\cos x}{\sin x}\right)}$$

$\cos x$ ✓

$$\frac{1 - \sin^2 x}{\cos x}$$

$$\frac{\cos^2 x}{\cancel{\cos x}}$$

$\cos x$ ✓

c) $\dfrac{\sin 2A}{2-2\cos^2 A} = \cot A$

d) $\dfrac{1-\cos x}{\sin x} = \dfrac{1}{\csc x + \cot x}$

e) $\dfrac{\sec x}{1-\cos x} = \dfrac{\sec x + 1}{\sin^2 x}$

f) $\dfrac{\cos 2x}{\cos x - \sin x} = \dfrac{\sin 2x}{2\cos x} + \cos x$