6.1 Reciprocal, Quotient and Pythagorean Identities

$3x^2 + 8 = 20$ is an example of an equation. It is true for only certain values of the variable *x*. In this case the solutions are $x = \pm 2$.

Can the expression $3x^2 + 8$ always be replaced with 20? no, only if $x = \pm 2$ $2x+1=9$

Can 20 always be replaced with $3x^2 + 8$? no, only when $x = \pm 2$

$(x-3)^2 = x^2 - 6x + 9$ is an example of an identity. It is true for all permissible values of the variable *x*.

Can $(x-3)^2$ always be replaced with $x^2 - 6x + 9$? yes $x + x = 2x$

Can $x^2 - 6x + 9$ always be replaced with $(x-3)^2$? yes

Some other examples are:

Equation	***Identity***
$\sin x = 0.5$	$\sin^2 x + \cos^2 x = 1$
$5\tan x + 3 = 8$	$\tan x = \dfrac{\sin x}{\cos x}$
$\csc x = 2$	$\csc x = \dfrac{1}{\sin x}$

true for all permissible values.

NPV $\cos x \neq 0$

$\sin x \neq 0$

We currently have the following identities

Reciprocal Identities	$\sec\theta = \dfrac{1}{\cos\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$
Quotient Identities	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	

pick a random number and see if both sides are equal.

Potential identities can be verified numerically or graphically.

graph both sides and see if the graphs are identical.

Example 1.

a) Determine the non-permissible values in the equation: $\csc\theta = \frac{\cot\theta}{\cos\theta}$.

$$\csc\theta = \frac{\left(\frac{\cos\theta}{\sin\theta}\right)}{\cos\theta}$$

$$\boxed{\begin{matrix}\sin\theta \neq 0 \\ \cos\theta \neq 0\end{matrix}}$$

b) Numerically verify that $\theta = 45^0$ and $\theta = \frac{\pi}{6}$ are solutions to this equation.

$\theta = 45^0$			$\theta = \frac{\pi}{6}$	
$\csc\theta$	=	$\frac{\cot\theta}{\cos\theta}$	$\csc\theta$	$\frac{\cot\theta}{\cos\theta}$
$\csc(45^\circ)$		$\frac{\cot(45^\circ)}{\cos(45^\circ)}$	$\csc\left(\frac{\pi}{6}\right)$	$\frac{\cot\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}$
$\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}$	=	$\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}$ ✓	$\frac{2}{1}$	$\frac{\frac{\sqrt{3}}{1}}{\frac{\sqrt{3}}{2}}$
				$\frac{\cancel{\sqrt{3}}}{1} \times \frac{2}{\cancel{\sqrt{3}}} = \frac{2}{1}$

c) Use your calculator and graph over the domain $-2\pi \le \theta \le 2\pi$ to decide whether this is an identity.

$$Y_1 = \frac{1}{\sin x}$$

$$Y_2 = \frac{\left(\frac{1}{\tan(x)}\right)}{\cos(x)}$$

the 2 graphs do overlap. for $-2\pi \le x \le 2\pi$

verifying numerically and graphically are pretty good indicators of identities, but the only definitive proof is algebraic

Example 2.

a) Determine the non-permissible values, in radians, of the variable in each expression.

b) Simplify the expression.

i) $\cos x \tan x$ NPV: $\cos x \neq 0$

$$\cancel{\cos x}\cdot\frac{\sin x}{\cancel{\cos x}}$$

$$\sin x$$

ii) $\dfrac{\tan x}{\sin x}$ NPV: $\cos x \neq 0,\ \sin x \neq 0$

$$\frac{\frac{\sin x}{\cos x}}{\sin x}$$

$$\frac{\cancel{\sin x}}{\cos x}\cdot\frac{1}{\cancel{\sin x}} = \frac{1}{\cos x}$$

$$\frac{\frac{\sin x}{\cos x}}{\sin x}\times\frac{\cos x}{\cos x}$$

$$\frac{\cancel{\sin x}}{\cancel{\sin x}\cos x}$$

$$\frac{1}{\cos x}$$

iii) $\dfrac{\tan x}{\sec x \sin x}$ $\cos x \neq 0,\ \sin x \neq 0$

$$\frac{\left(\frac{\sin x}{\cos x}\right)}{\frac{1}{\cos x}\cdot\sin x}$$

$$\frac{\left(\frac{\sin x}{\cancel{\cos x}}\right)\cdot\cancel{\cos x}}{\left(\frac{\sin x}{\cancel{\cos x}}\right)\cdot\cos x} = 1$$

iv) $\csc x \tan x \sin^2 x$ NPV: $\sin x \neq 0,\ \cos x \neq 0$

$$= \frac{1}{\cancel{\sin x}}\cdot\frac{\cancel{\sin x}}{\cos x}\cdot\sin^2 x$$

$$= \frac{\sin^2 x}{\cos x}$$

note: $\dfrac{\sin x\cdot\sin x}{\cos x}$

$$= \frac{\sin x}{\cos x}\cdot\sin x$$

$$= \tan x\cdot\sin x$$

Pythagorean Identity:

On the unit circle

$$x^2 + y^2 = 1$$

therefore

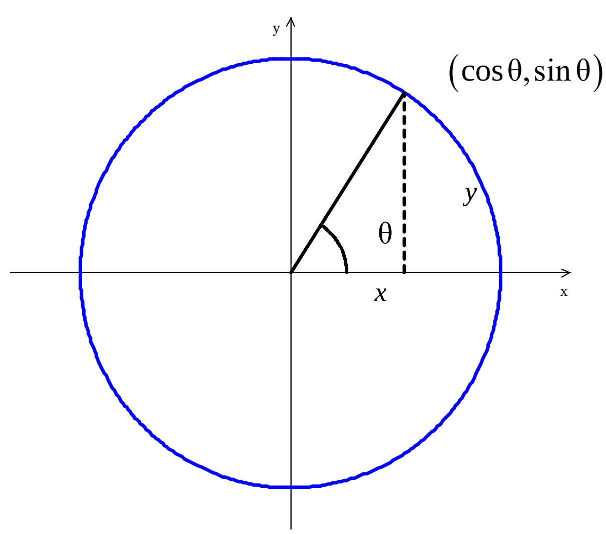
very important

$$\cos^2\theta + \sin^2\theta = 1$$

Note: This also means

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\sin^2\theta = 1 - \cos^2\theta$$



Example 3.

a) Verify that the equation $\tan^2\theta + 1 = \sec^2\theta$ is true when $\theta = \frac{\pi}{3}$

$$\left(\frac{\sqrt{3}}{1}\right)^2 + 1 = \left(\frac{2}{1}\right)^2$$

$$3 + 1 = 4 \quad \checkmark \text{ true for } \theta = \frac{\pi}{3}$$

b) Multiply the both sides of the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$ by $\frac{1}{\cos^2\theta}$. Why does this create a new identity?

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

c) Multiply the both sides of the Pythagorean Identity $\cos^2\theta + \sin^2\theta = 1$ by $\frac{1}{\sin^2\theta}$. What new identity does this create?

$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

Note that the Pythagorean identities can each be rearranged to produce other identities:

$\sin^2 x + \cos^2 x = 1$	$\tan^2 x + 1 = \sec^2 x$	$\cot^2 x + 1 = \csc^2 x$
$\sin^2 x = 1 - \cos^2 x$	$\tan^2 x = \sec^2 x - 1$	$\cot^2 x = \csc^2 x - 1$
$\cos^2 x = 1 - \sin^2 x$ $\cos^2 x = (1+\sin x)(1-\sin x)$	$1 = \sec^2 x - \tan^2 x$ $1 = (\sec x + \tan x)(\sec x - \tan x)$	$1 = \csc^2 x - \cot^2 x$

It is also important to note that the right hand side in each of the statements in the bottom two rows is a **difference of squares**, and can therefore be factored.

Thus, $\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$

Also, these identities are true regardless of whether you are in degrees or radians.

ie. $\sin^2 .47 + \cos^2 .47 = 1$ $\quad$ $\sin^2 59^0 + \cos^2 59^0 = 1$ $\quad$ $\sin^2 \frac{2\pi}{7} + \cos^2 \frac{2\pi}{7} = 1$