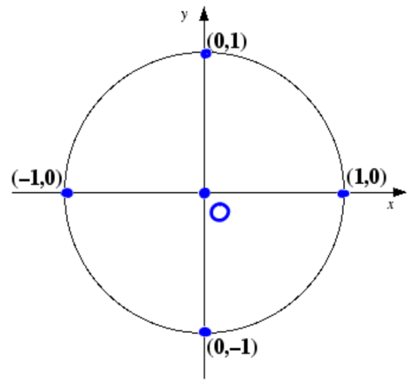
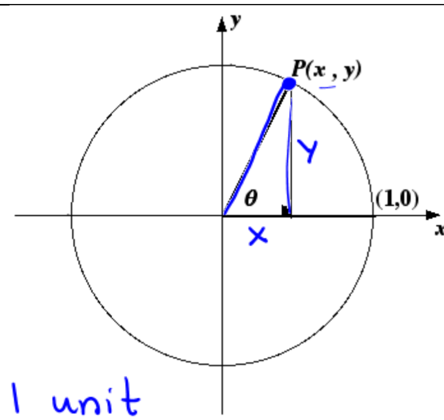
4.2A The Unit Circle

The unit circle is simply any circle whose radius equals _1___unit.

For our purposes, we will place the centre of a unit circle at the origin of the Cartesian plane.



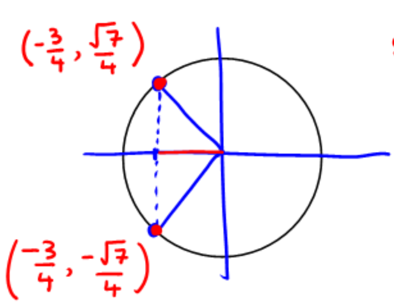
To find the equation of the unit circle, we can use Pythagoras Theorem. $x^2+y^2=1$ radius =	 1 unit

Example 1.

a) If the radius of the circle was 3 instead of 1, what would the equation be? $x^2+y^2=3^2$	b) If the radius of the circle was *r*, what would the equation be? $x^2+y^2=r^2$

Example 2. Determine the coordinates for all points on the unit circle which meet the stated conditions. In each case draw a diagram.

a) The *x*-coordinate is $-\dfrac{3}{4}$



$y^2+\left(-\frac{3}{4}\right)^2=1^2$

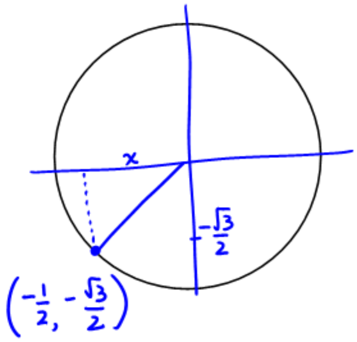
$y^2+\frac{9}{16}=1$

$y^2=\frac{7}{16}$

$y=\pm\frac{\sqrt{7}}{\sqrt{16}}$

$y=\pm\frac{\sqrt{7}}{4}$

b) The *y*-coordinate is $-\frac{\sqrt{3}}{2}$ and the point is in quadrant III



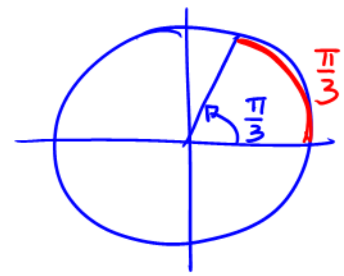
$x^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = 1$

$x^2 + \frac{3}{4} = 1$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

$x = \pm\sqrt{\frac{1}{4}}$



$a = r\theta$

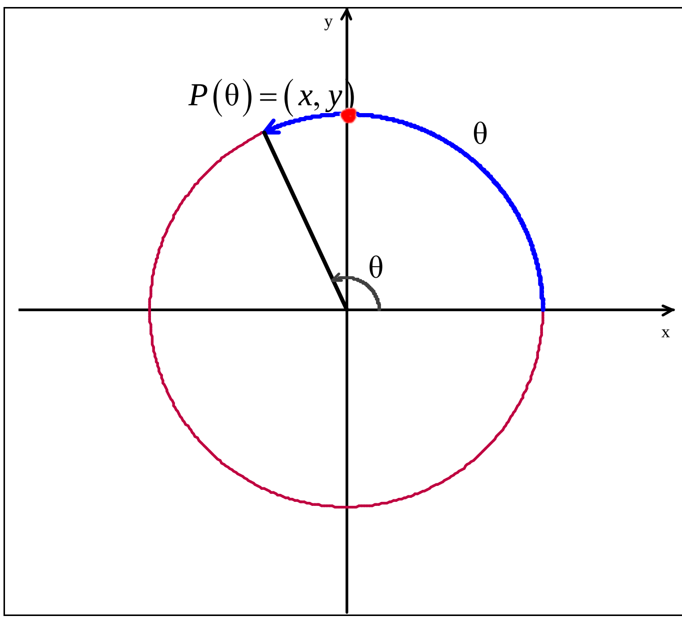
Arc Length and Angle Measure

$a = \theta$

We have already seen that the formula $a = \theta r$ can be used to determine the arc length when the central angle is θ (in radians) and the radius is *r*. In the case of a unit circle, the radius is 1, and this formula then says $a = \theta(1)$ or more simply $a = \theta$. This means that in a unit circle, a central angle of θ is subtended by an arc with length θ (except that the arc length is measured in length units)

Thus on the unit circle, if the central angle is .63, then the subtended arc has a length of .63; if the central angle is $\frac{\pi}{6}$, then the arc has a length of $\frac{\pi}{6}$ (Note that these statements are only true if the angle is in radians)

Define a function $P(\theta) = (x, y)$ which relates the angle θ (or arclength θ) with the coordinates (x, y) on the unit circle



$P\left(\frac{\pi}{2}\right)$ means to travel an arclength of $\frac{\pi}{2}$ on the unit circle and state the coordinates of the point of intersection of the terminal arm and the circle.
Therefore, $P\left(\frac{\pi}{2}\right) =$ __(0, 1)__

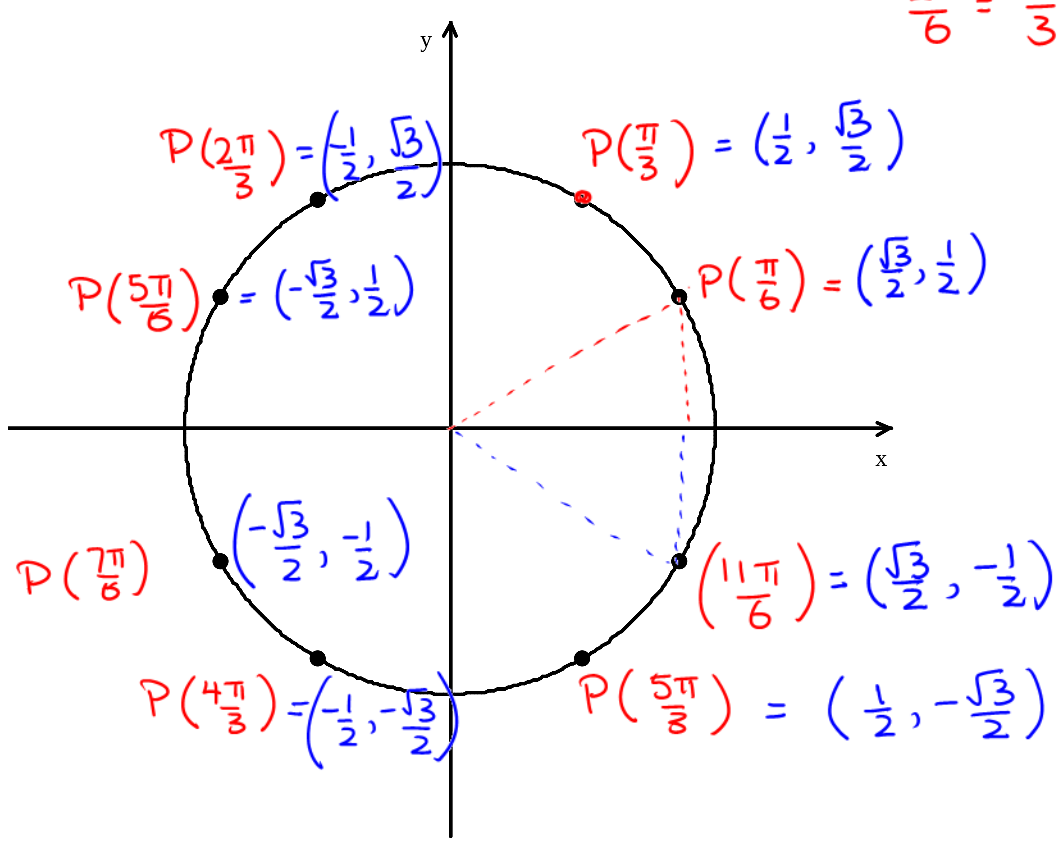
$P(\pi) =$ __(−1, 0)__

$P\left(\frac{3\pi}{2}\right) =$ __(0, −1)__

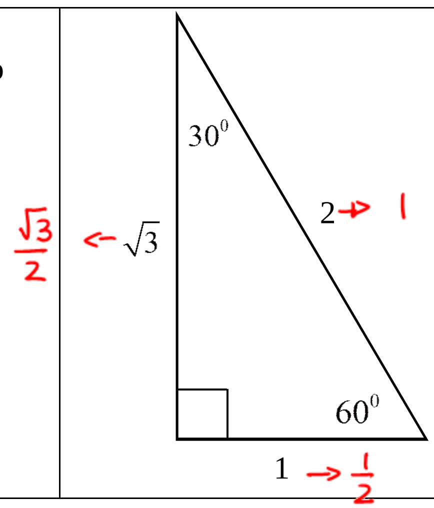
Example 3.

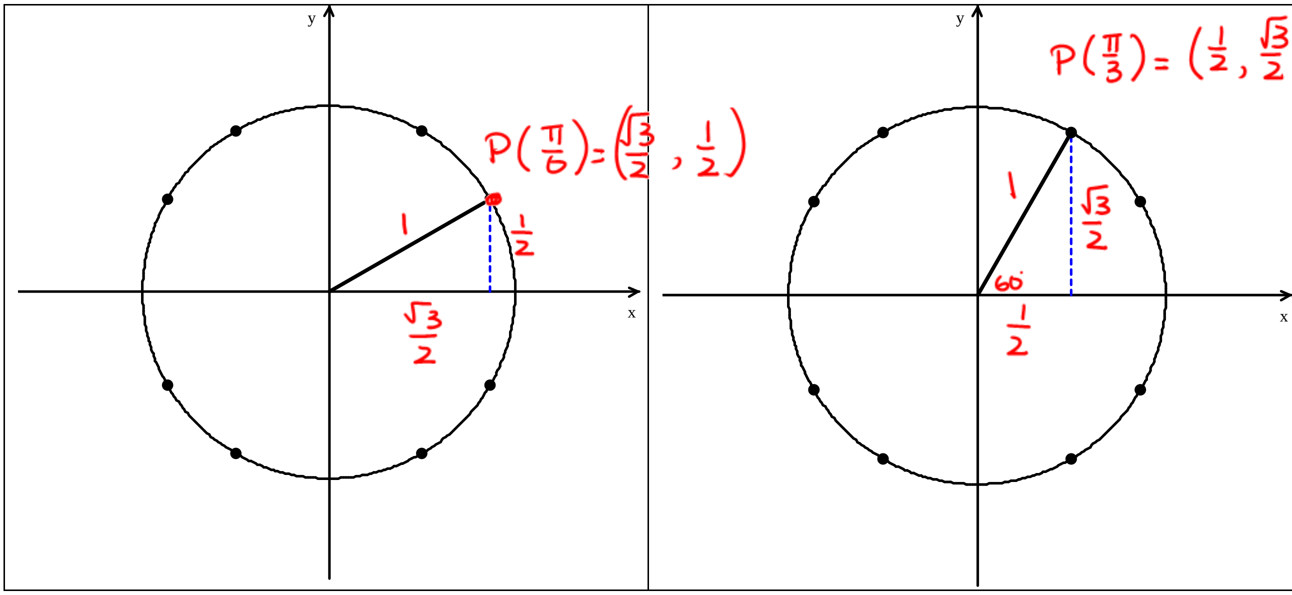
a) On a diagram of a unit circle, show the integral multiples of $\frac{\pi}{6}$ in the interval $0 \le \theta \le 2\pi$

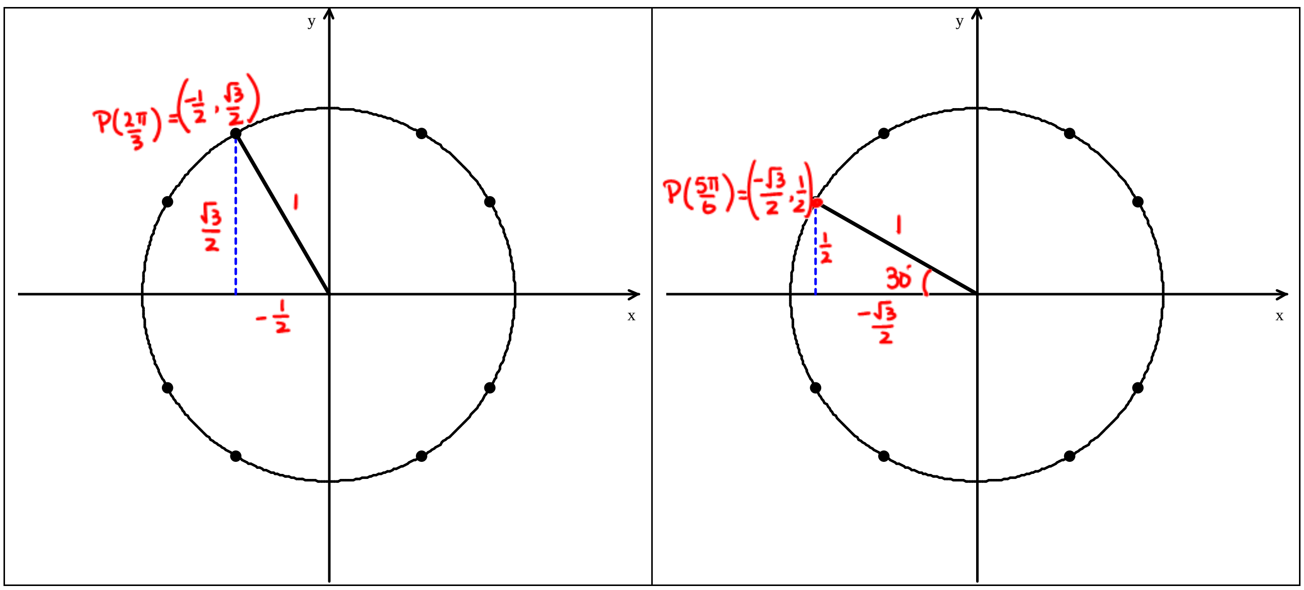
$\frac{2\pi}{6} = \frac{\pi}{3}$



b) What are the coordinates of each point $P(\theta)$?

To do this we need to recall properties of a $30^0 - 60^0 - 90^0$ triangle. Convert this triangle to one with a hypotenuse of length one.	





What patterns do you observe in the completed diagram on the previous page?

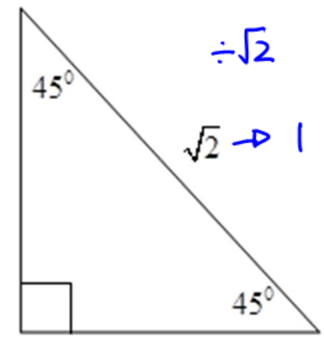
all angles with the same reference angle have the same (x, y) but the signs are different

eg $P\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

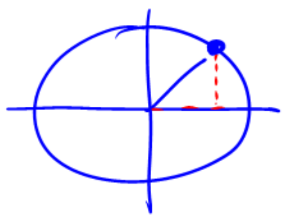
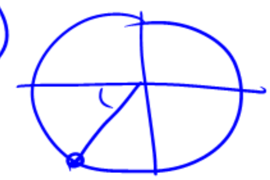
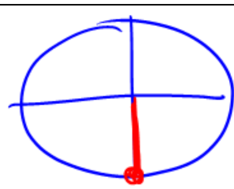
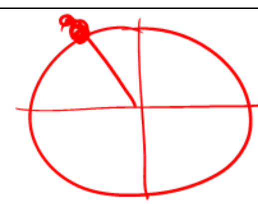
$P\left(\frac{5\pi}{6}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$P\left(\frac{7\pi}{6}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$P\left(\frac{11\pi}{6}\right) = \left(+\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Recall also the properties of a $45^0 - 45^0 - 90^0$ triangle. Convert this triangle to one with a hypotenuse of length one. note $\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}}$ ← 1 

Example 4. Determine: $\frac{1}{\sqrt{2}}$ can also be written as $\frac{\sqrt{2}}{2}$

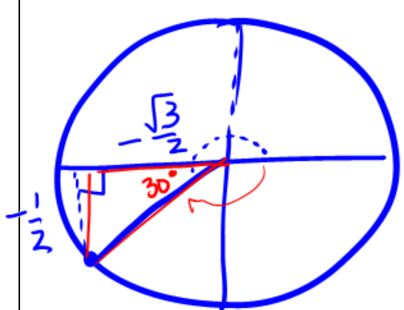
a) $P\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ 	b) $P\left(-\frac{3\pi}{4}\right) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$  
c) $P\left(\frac{7\pi}{2}\right) = 3\frac{1}{2}\pi$ 1 full rotation $\frac{3\pi}{2}$  $P\left(\frac{7\pi}{2}\right) = (0, -1)$	d) $P\left(-\frac{4\pi}{3}\right)$ $\theta_R = \frac{\pi}{3}$  $P\left(-\frac{4\pi}{3}\right) = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ p187 #1-4, 10, 11

4.2B The Unit Circle – Given the point, finding the angle

The function $P(\theta)=(x,y)$ gives the coordinates of the point on the unit circle associated with a nagle of rotation (or arclength) of θ. Thus $P\left(-\frac{3\pi}{4}\right)=\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ relates the point $\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ with the angle (or arclength) of $-\frac{3\pi}{4}$.

Example 1. Identify a measure for the central angle θ in the interval $0\le\theta<2\pi$ such that $P(\theta)$ is the given point:

<table>
<tr>
<td>
a) $\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

Draw the unit circle and label the diagram in an appropriate way. Use your knowledge of special triangles to determine the value of θ.


$\theta = 180^\circ + 30^\circ$
$= 210^\circ$
$\theta = \pi + \frac{\pi}{6}$
$\theta = \frac{7\pi}{6}$
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This question is really asking what value of θ makes $P(\theta) = \left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$? Or what rotation angle brings you to the point $\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$?

How many answers does this question have?
infinite because we can consider full rotations
Why has the interval been limited to $0\le\theta<2\pi$?
- you find only 1 answer and the rest are found by adding $n2\pi$ to it.
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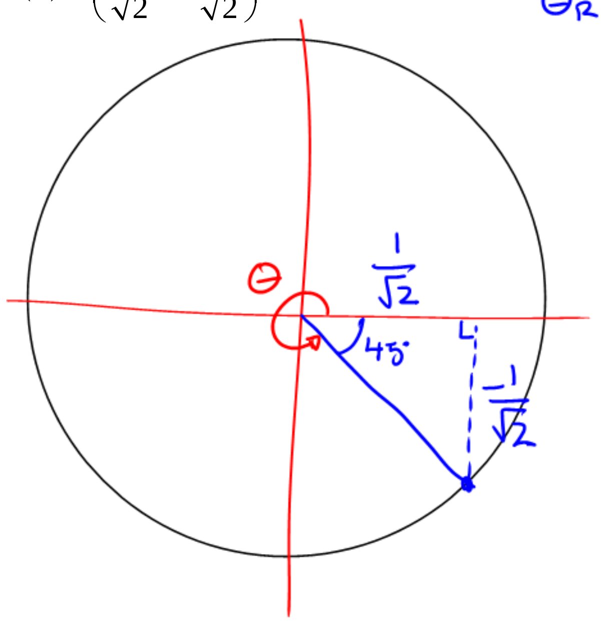
b) $P(\theta)=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$



$\theta_R = 45^\circ$ or $\frac{\pi}{4}$

$\theta = 360^\circ - 45^\circ$ or $\theta = 2\pi - \frac{\pi}{4}$

$= 315^\circ$ $\quad$ $= \frac{7\pi}{4}$

no restriction answer

$\theta = 315^\circ + n(360^\circ)$

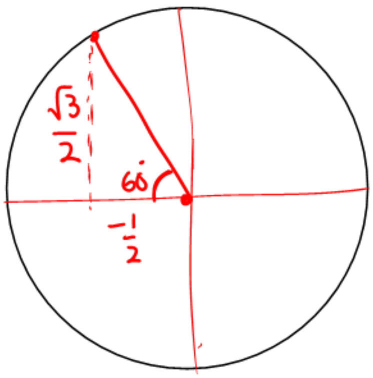
or

$\theta = \frac{7\pi}{4} + n(2\pi)$

n is any integer

6π or 3 full rotations apart

Example 2. Determine all values for θ in the interval $-\pi \le \theta < 5\pi$ such that $P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



$\theta = 180 - 60 = 120°$ or $\theta = \pi - \frac{\pi}{3}$

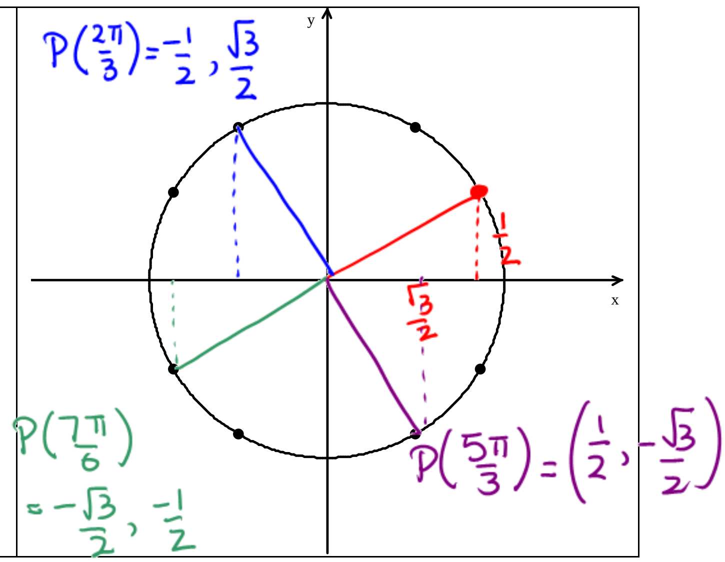
$= \frac{2\pi}{3}$ ←

$\frac{8\pi}{3}$, $\frac{14\pi}{3}$

$\theta = \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}$

Example 3. What is the relationship between the points that are $\frac{1}{4}$ rotation apart on the unit circle?

Start with the point $P\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Move $\frac{1}{4}$ rotation from this point. Determine this new point and its coordinates. Repeat this process by moving $-\frac{1}{4}$ rotation from the original point.



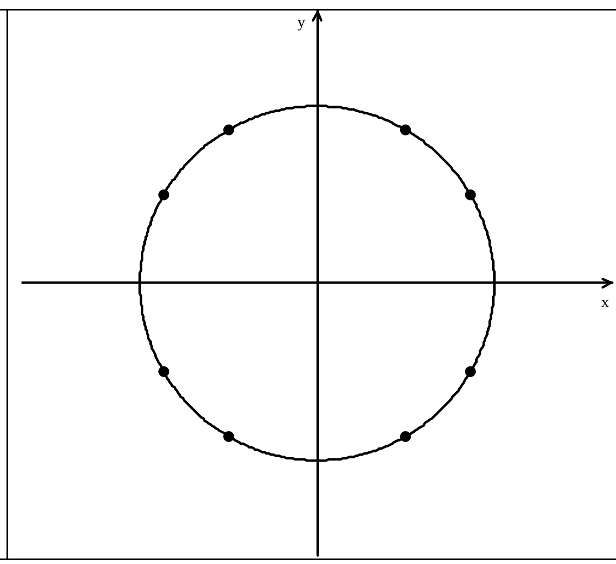
Repeat these steps with the point $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$P\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$P\left(\frac{\pi}{6} + \frac{\pi}{2}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$P\left(\frac{\pi}{6} + 2\frac{\pi}{2}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

$P\left(\frac{\pi}{6} + \frac{3\pi}{2}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$



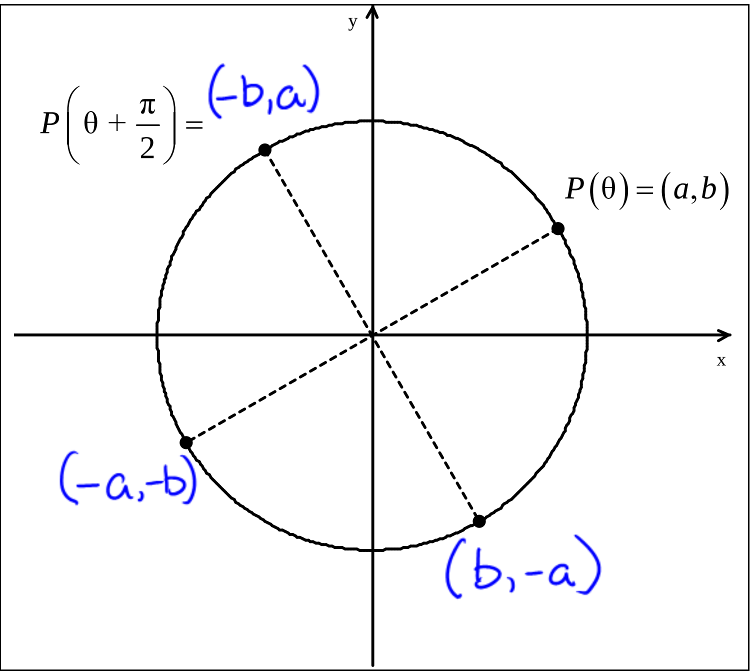
Summarize your findings in the diagram to the right.

$P(\theta) = (a,b)$

$P\left(\theta + \frac{\pi}{2}\right) = (-b,a)$

$P\left(\theta + \frac{2\pi}{2}\right) = (-a,-b)$

$P\left(\theta + \frac{3\pi}{2}\right) = (b,-a)$



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4.2b p 187 # 5-7, 9, 12, 13, 15
C1, C2.