3.2a Long Division of Polynomials

In each of the following long division statements, identify the divisor, dividend, quotient and remainder.

$$\begin{array}{r l}
 & 23 \leftarrow \text{Quotient} \\
\text{Divisor} \longrightarrow 14 & \overline{)328} \leftarrow \text{Dividend} \\
 & \underline{28} \\
 & 48 \\
 & \underline{42} \\
 & 6 \leftarrow \text{Remainder.}
\end{array}$$

$$\begin{array}{r l}
 & 3x+2 \\
x+2 & \overline{)3x^2+8x-5} \\
 & \underline{3x^2+6x} \\
 & 2x-5 \\
 & \underline{2x+4} \\
 & -9
\end{array}$$

What is the relationship between the divisor, dividend, quotient and remainder?

Dividend = Divisor × Quotient + Rem

Dividing a polynomial by a binomial of the form $x-a$

Example 1. Divide the polynomial $P(x)=2x^3-4x+5x^2-6$ by $x-3$

Before doing the division, arrange the powers in descending order. Additionally, each power less than the degree must be included in the dividend – this means that you might have to include terms with a coefficient of zero.

$$\begin{array}{r l}
 & 2x^2+11x+29 \\
x-3 & \overline{)2x^3+5x^2-4x-6} \\
 & \underline{2x^3-6x^2} \\
 & 11x^2-4x \\
 & \underline{11x^2-33x} \\
 & 29x-6 \\
 & \underline{29x-87} \\
 & 81
\end{array}$$

Divide $2x^3$ by x to get $\underline{2x^2}$
Multiply $x-3$ by $\underline{2x^2}$ and subtract
Divide $\underline{11x^2}$ by x to get $\underline{11x}$
Multiply $x-3$ by $\underline{11x}$ and subtract
Divide $\underline{29x}$ by x to get $\underline{29}$
Multiply $x-3$ by $\underline{29}$ and subtract

Express the result in the form $\dfrac{P(x)}{x-a}=Q(x)+\dfrac{R}{x-a}$.

$$\frac{2x^3+5x^2-4x-6}{x-3}=\left(2x^2+11x+29\right)+\frac{81}{x-3}$$

What restrictions are there on the variable? $x \neq 3$

Write a statement that can be used to check the division. Then check your answer.

$$2x^3 + 5x^2 - 4x - 6 = (x-3)(2x^2 + 11x + 29) + 81$$

$$= 2x^3 + 11x^2 + 29x - 6x^2 - 33x - 87 + 81$$

$$= 2x^3 + 5x^2 - 4x - 6$$

Another way to check your answer is to substitute a value for *x*. Check the division by substituting $x = 4$

Try $x = 0$

$$2(0)^3 + 5(0)^2 - 4(0) - 6 \stackrel{?}{=} (0-3)(2(0)^2 + 11(0) + 29) + 81$$

$$-6 \stackrel{?}{=} -3(29) + 81$$

$$-6 = -6 \checkmark$$

Example 2. Divide the polynomial $P(x) = 3x^4 - 4x^3 + 7x - 5$ by $x - 2$.

$$
\begin{array}{r}
3x^3 + 2x^2 + 4x + 15 \\
x-2 \overline{\smash{\big)}\, 3x^4 - 4x^3 + 0x^2 + 7x - 5} \\
\underline{3x^4 - 6x^3} \qquad\qquad\qquad\quad \\
2x^3 + 0x^2 \qquad\qquad \\
\underline{2x^3 - 4x^2} \qquad\qquad \\
4x^2 + 7x \qquad \\
\underline{4x^2 - 8x} \qquad \\
15x - 5 \\
\underline{15x - 30} \\
25
\end{array}
$$

Express the result in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$.

$$\frac{3x^4 - 4x^3 + 7x - 5}{x-2} = 3x^3 + 2x^2 + 4x + 15 + \frac{25}{x-2}$$

What restrictions are there on the variable?

$x \neq 2$

Check your answer.

$x = 0$

$$\frac{-5}{-2} \stackrel{?}{=} 15 + \frac{25}{-2}$$

$$\frac{5}{2} \stackrel{\checkmark}{=} \frac{5}{2}$$

Example 3. The volume of a box is given by the function $V(x) = 2x^3 + 9x^2 + 7x - 6$. If the height of the box is $x+3$, determine the other dimensions in terms of x.

$$\begin{array}{r} 2x^2+3x-2 \\ x+3\,\overline{)\,2x^3+9x^2+7x-6} \\ \underline{2x^3+6x^2} \\ 3x^2+7x \\ \underline{3x^2+9x} \\ -2x-6 \\ \underline{-2x-6} \\ 0 \end{array}$$

Because $x+3$ is the height, $2x^2+3x-2$ must be the area of the base.

Factor $(2x^2+3x-2) = (2x-1)(x+2)$ Other dimensions are $(x+2)$ & $(2x-1)$

Synthetic Division. This is an alternate way of dividing a polynomial by $x-a$.

Example 4. Divide the polynomial $P(x) = 3x^4 - 4x^3 + 7x - 5$ by $x-2$ using synthetic division.

Dividing by $x-2$, so place whatever makes this equal to 0 here

	$3x^4$	$-\,4x^3$	$+\,0x^2$	$+\,7x$	$-\,5$
	↓	↓	↓	↓	↓
2	3	–4	0	7	–5
+	↓	6	4	8	30
	3	2	4	15	25

Rem

Bring the 3 down, and then successively multiply and add

$Q(x) = 3x^3 + 2x^2 + 4x + 15$

Rem = 25

Compare this result with what was done in example 2.

Example 5. Redo example 3 using synthetic division.

$$\begin{array}{r|rrrr} -3 & 2 & 9 & 7 & -6 \\ & & -6 & -9 & 6 \\ \hline & \underbrace{2 \quad 3 \quad -2} & & & \boxed{0} \end{array}$$

Quotient = Base area
$= 2x^2+3x-2$

Base area $= 2x^2+3x-2$
$= (2x-1)(x+2)$

Other dimensions are $2x-1$ and $x+2$

Example 6. Use synthetic division to determine the remainder when $x^4 - 16$ is divided by $i)\ x+1$ $ii)\ x+2$ $iii)\ x+4$.

-1	1	0	0	0	-16
		-1	1	-1	1
	1	-1	1	-1	-15

Rem $= -15$

-2	1	0	0	0	-16
+		-2	4	-8	16
	1	-2	4	-8	$\boxed{0}$

Rem $= 0$

-4	1	0	0	0	-16
		-4	16	-64	256
	1	-4	16	-64	240

Rem $= 240$

What information does this provide you with?

$x+2$ is a factor of $x^4 - 16$ (so is $x^3 - 2x^2 + 4x - 8$)

and $x+1$ and $x+4$ are not factors.

Example 7. When the polynomial $x^3 - 5x^2 + kx - 7$ is divided by $x+5$ the remainder is 3. What must the value of k be?

-5	1	-5	k	-7
	$\downarrow$	-5	50	$-5k-250$
	1	-10	$k+50$	$-5k-257$

$-5k - 257 = \cancel{6}3$

$-5k = 260$

$k = \frac{260}{-5} = -52$

It is important to note that synthetic division, although faster than long division, should only be used when dividing by a binomial of the form $x - a$ (ie leading coefficient of one). Long division however can be easily adjusted to accommodate division by more complicated polynomials.

When dividing whole numbers, the remainder must always be a positive number that is less than the divisor. When dividing polynomials, the degree of the remainder must always be less than the degree of the divisor. When you are dividing by a linear binomial, this means that the remainder must always have a degree of ____0____ or in other words, it must be a ____constant____ .