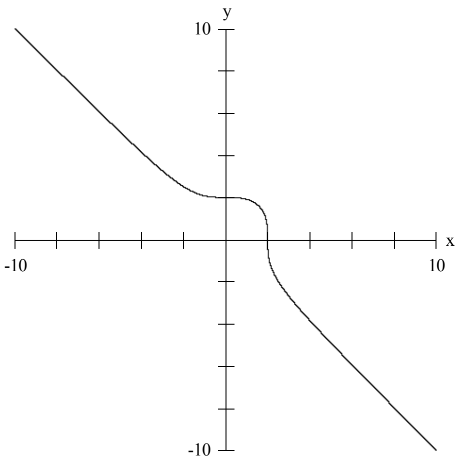
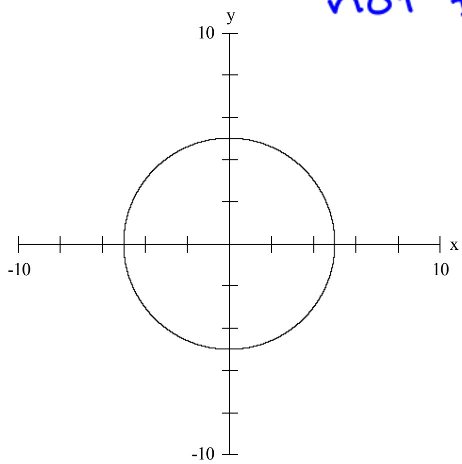
Implicit Differentiation

Some functions are defined explicitly: they may be written in the form $y = f(x)$. Other functions may be defined implicitly – it may not be possible to express them in the form $y = f(x)$ (ie. It may not be possible to solve for *y* in terms of *x*). Although these are sometimes called functions, they are really relations, and may be viewed as the union of several functions

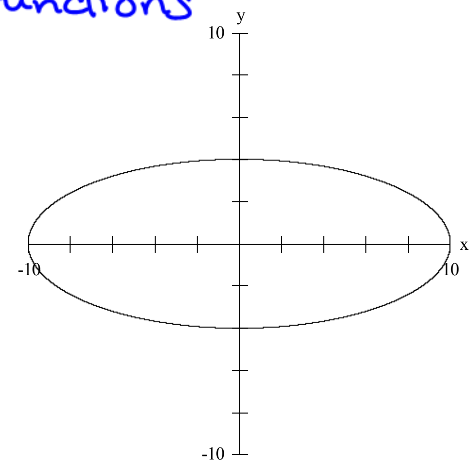
none of these passes the vertical line test, not functions



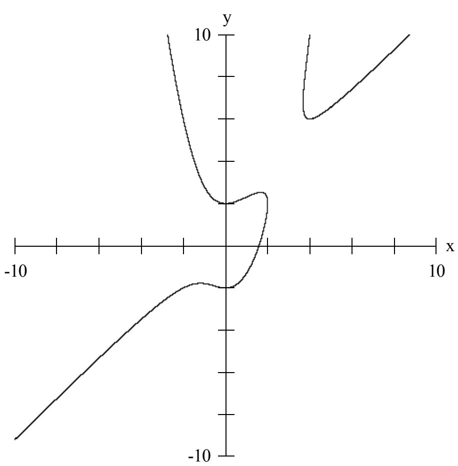
$x^3 + y^3 = 8$



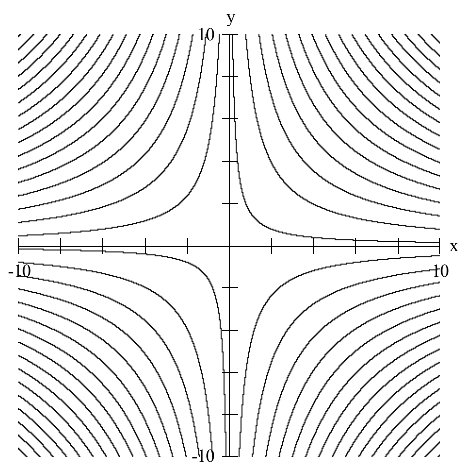
$x^2 + y^2 = 25$



$4x^2 + 25y^2 = 400$



$x^3 - x^2y + y^2 = 4$

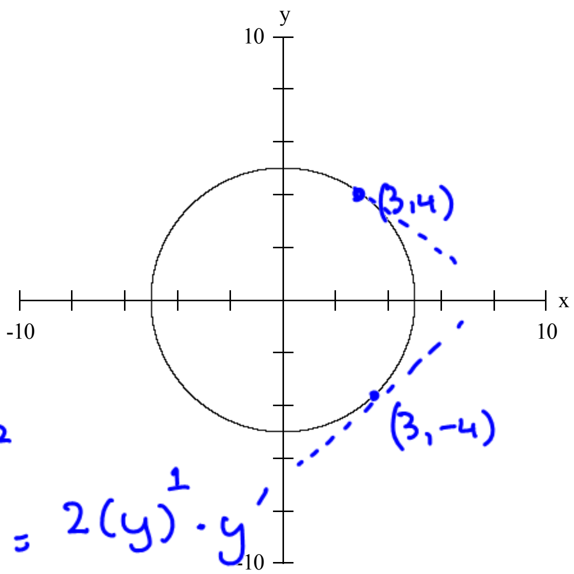


$\sin(xy) = 1$

For any point on each of the above curves, it still makes sense to talk about the slope of the tangent at that point, and hence also makes sense to talk about the derivative.

To find the derivative of an implicitly defined function, we use a technique called ***implicit differentiation***. Essentially you differentiate both sides of the equation with respect to *x* (this usually involves the chain rule, plus the product of quotient rules) and then solve the resulting equation for $\frac{dy}{dx}$. Generally the derivative will be in terms of *x* and *y* rather than just in terms of *x*.

Example: Determine the derivative at any point on the circle $x^2 + y^2 = 25$.



$$\frac{d(x^2+y^2)}{dx} = \frac{d(25)}{dx}$$

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = 0$$

$$2x + 2(y)^1 \cdot \frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

$$\frac{d(y)^2}{dx} = 2(y)^1 \cdot y'$$

a) What is the equation of the tangent to the curve at $(3,4)$?

$$\frac{dy}{dx} = \frac{-x}{y}\bigg|_{(x,y)=(3,4)} = -\frac{3}{4}$$

$$\boxed{(y-4) = \frac{-3}{4}(x-3)}$$

b) How would you expect the slope at $(3,4)$ to compare with the slope at $(3,-4)$?

$$\frac{dy}{dx} = \frac{-x}{y}\bigg|_{(x,y)=(3,-4)} = \frac{-3}{(-4)} = \frac{3}{4}$$

c) What would you expect the derivative at $x = 5$ to be? What does this say about the tangent line?

$$\frac{dy}{dx} = -\frac{x}{y}\bigg|_{(x,y)=(5,0)} = -\frac{5}{0} = \text{undefined}$$

or a vertical tangent line.

Implicit Differentiation

Find $\frac{dy}{dx}$ for each of the following:

$xy = 8$

$$\frac{d(x \cdot y)}{dx} = \frac{d(8)}{dx}$$

$$1 \cdot y + y' \cdot x = 0$$

$$y'x = -y$$

$$y' = \frac{-y}{x}$$

$$= \frac{\frac{-8}{x}}{x} = \frac{-8}{x^2}$$

$$y = \frac{8}{x}$$

$$y = 8 \cdot x^{-1}$$

$$y' = -8x^{-2}$$

2. $x^3 y = 18$

$$\frac{d(x^3 y)}{dx} = \frac{d(18)}{dx}$$

$$3x^2 \cdot y + y' \cdot x^3 = 0$$

$$y'x^3 = -3x^2 y$$

$$y' = \frac{-3x^2 y}{x^3}$$

$$y' = \frac{-3y}{x}$$

3. $x y^3 = x$

$$\frac{d(xy^3)}{dx} = \frac{d(x)}{dx}$$

$$(1)(y^3) + (x)(3y^2)(y') = 1$$

$$3xy^2 \cdot y' = 1 - y^3$$

$$y' = \frac{1 - y^3}{3xy^2}$$

4. $x y^3 = x$

5. $x = \sin y$

$$\frac{d(x)}{dx} = \frac{d(\sin y)}{dx}$$

$$1 = [\cos y]\, y'$$

$$y' = \frac{1}{\cos y}$$

6. $\tan(x y) = 1$

$$\frac{d(\tan(x \cdot y))}{dx} = \frac{d(1)}{dx}$$

$$\frac{\sec^2(x \cdot y)\left[x \cdot y' + y(1)\right]}{\sec^2(xy)} = \frac{0}{\sec^2(xy)}$$

$$x \cdot y' + y = 0$$

$$xy' = -y$$

$$\boxed{y' = -\frac{y}{x}}$$

7. $x^2 + xy + y^2 = 12$

$$\frac{d(x^2)}{dx} + \frac{d(xy)}{dx} + \frac{d(y^2)}{dx} = \frac{d(12)}{dx}$$

$$2x + [(1)(y) + y'(x)] + 2y \cdot y' = 0$$

$$y' \cdot (x) + y' \cdot (2y) = -2x - y$$

$$y'(x+2y) = -2x - y$$

$$\boxed{y' = \frac{-2x-y}{x+2y}}$$

8. Find $\frac{dy}{dx}$ for $x^2 y + y = 4$ by a) using implicit differentiation

b) solving for y and then differentiating as usual

$$\frac{d(x^2y)}{dx} + \frac{d(y)}{dx} = \frac{d(4)}{dx}$$

$$2xy + x^2 \cdot y' + y' = 0$$

$$x^2 y' + y' = -2xy$$

$$y'(x^2+1) = -2xy$$

$$y' = \frac{-2xy}{x^2+1}$$

$$y(x^2+1) = 4$$

$$y = \frac{4}{x^2+1}$$

$$y = 4(x^2+1)^{-1}$$

$$y' = -4(x^2+1)^{-2}(2x)$$

$$y' = \frac{-8x}{(x^2+1)^2}$$

$$y' = \frac{-2x\left(\frac{4}{x^2+1}\right)}{x^2+1} = -2x \cdot \left(\frac{4}{x^2+1}\right) \cdot \frac{1}{x^2+1}$$

$$= \frac{-8x}{(x^2+1)^2}$$

9. Find the tangents and normals to the curve $x^2 + 4y^2 = 17$ when $x = 1$

① Find y-coordinate.

$$(1)^2 + 4y^2 = 17$$

$$4y^2 = 16$$

$$y^2 = 4$$

$$y = \pm 2$$

$$\frac{d(x^2)}{dx} + \frac{d(4y^2)}{dx} = \frac{d(17)}{dx}$$

$$2x + 8y \cdot y' = 0$$

$$8y \cdot y' = -2x$$

$$y' = \frac{-2x}{8y}$$

(1,2)

$$y' = \frac{-2(1)}{8(2)}$$

$$y' = -\frac{1}{8}$$

(1,-2)

$$y' = \frac{-2(1)}{8(-2)}$$

$$y' = \frac{1}{8}$$

tangents

$$y - 2 = -\frac{1}{8}(x-1)$$

$$y + 2 = \frac{1}{8}(x-1)$$

normals

$$y - 2 = 8(x-1)$$

$$y + 2 = -8(x-1)$$

10. Find the points on the curve $x^2 - y^2 = -4$ where the tangent is parallel to the x-axis.

finding where $y' = 0$

$$\frac{d(x^2 - y^2)}{dx} = \frac{d(-4)}{dx}$$

$$2x - 2y \cdot y' = 0$$

$$-2y \cdot y' = -2x$$

$$y' = \frac{-2x}{-2y}$$

$$\boxed{y' = \frac{x}{y}}$$

$$\boxed{x = 0}$$

sub into initial equation

$$(0)^2 - y^2 = -4$$

$$-y^2 = -4$$

$$y^2 = 4$$

$$y = \pm 2$$

$\boxed{(0,2) \text{ and } (0,-2)}$

2.13 Review Warmup

1. Determine $\frac{dy}{dx}$ for:

   a) $y = \sin^3(5x+3)$

   b) $y = \sqrt{\sin^2 x + x^3}$

2. Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the relation: $xy = 2y - 3x^2$

3. Find the equation of the tangent(s) to the relation in #2 at the points where $y = 1$

4. If a position function $s(t)$ had the property that $v = \sqrt{s}$, show that acceleration must be constant.

5. If $y = m\sin pt + n\cos pt$ where *m*, *n*, and *p* are constants, then what is $\frac{d^2y}{dx^2}$?

Unit 2 Review Warmup

1. Determine $\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{6}+h\right)-\tan\frac{\pi}{6}}{h}$

2. A rock is thrown into the air form a cliff that is 10m high. For any time $t \geq 0$, its height is given by $h(t) = -5t^2 + 40t + 10$.

   a) What is the maximum height reached?

   b) What is the velocity when it hits the ground?

3. Determine $\frac{dy}{dx}$ if $y = \cos(xy)$

4. Find the tangent to the curve $y^2x + x + y = 17$ at the point where $y = 2$

Unit 2 Review

Determine the derivative of the following functions

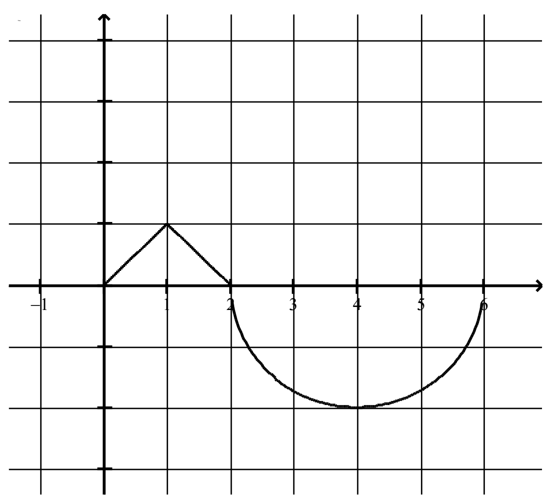
1. $f(x)=(4x+1)(1-x)^3$
2. $y=\dfrac{2-x}{3x+1}$
3. $y=\sqrt{3-2x}$
4. $y=\dfrac{2}{(5x+1)^3}$
5. $y=3x^{2/3}-4x^{1/2}-2$
6. $y=2\sqrt{x}-\dfrac{1}{2\sqrt{x}}$
7. $y=\sqrt{x^2+2x+1}$
8. $y=\dfrac{x}{\sqrt{1-x^2}}$
9. $y=\dfrac{1+x^2}{1-x^2}$

For questions 10-17, functions f and g have the values shown in the table

x	f	f'	g	g'
0	2	1	5	–4
1	3	2	3	–3
2	5	3	1	–2
3	10	4	0	–1

10. If $A=f+2g$, then $A'(3)=$
11. If $B=f\bullet g$ then $B'(2)=$
12. If $D=\dfrac{1}{g}$, then $D'(1)=$
13. If $H(x)=\sqrt{f(x)}$, then $H'(3)=$
14. If $K(x)=\dfrac{f(x)}{g(x)}$, then $K'(0)=$
15. If $M(x)=f(g(x))$, then $M'(1)=$
16. If $P(x)=f(x^3)$, then $P'(1)=$
17. If $S(x)=f^3(f(x))$, then $S'(0)=$

Use the graph to answer questions 18-20
The graph consists of two line segments and a semicircle.



18. $f'(x)=0$ for $x=$

19. $f'(x)$ does not exist for $x=$

20. $f'(5)=$

Determine $\frac{dy}{dx}$ for the following implicitly defined functions

21. $x^3 - xy + y^3 = 1$

22. $x + \cos(x+y) = 0$

23. $\sin x - \cos y - 2 = 0$

24. $3x^2 - 2xy + 5y^2 = 1$

25. If $f(x) = x^4 - 4x^3 + 4x^2 - 1$, for what values of x does graph have a horizontal tangent?

26. If $f(x) = 16\sqrt{x}$ then $f''(4) =$

27. If a point moves on the curve $x^2 + y^2 = 25$, then at $(0,5)$, $\frac{d^2y}{dx^2}$ is ____________

28. If $y = a\sin ct + b\cos ct$ where a, b, and c are constants, then $\frac{d^2y}{dt^2}$ is ____________

29. If $f(v) = \sin v$ and $v = g(x) = x^2 - 9$, then $(f \circ g)'(3) =$____________

30. If $f(x) = \frac{x}{(x-1)^2}$, then for what values of x does $f'(x)$ exist?

31. If $y = \sqrt{x^2+1}$, then what is the derivative of y^2 with respect to x^2?

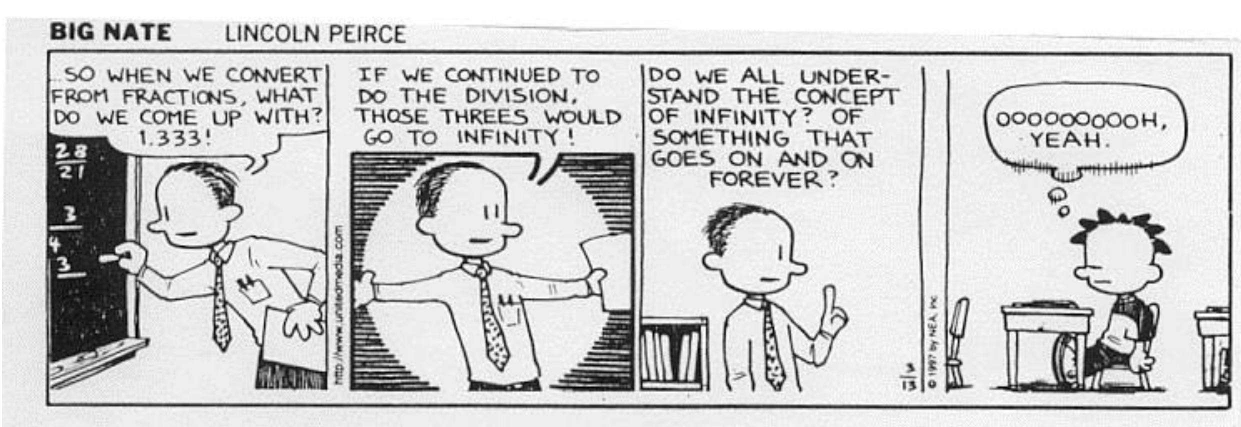
32. If $f(x) = \frac{1}{x^2+1}$ and $g(x) = \sqrt{x}$, then the derivative of $f(g(x))$ is ?

33. $\lim_{h\to 0}\frac{(1+h)^6 - 1}{h} =$ (Hint: Think about the definition of the derivative)

34. $\lim_{h\to 0}\frac{\sqrt[3]{8+h} - 2}{h} =$

35. $\lim_{x\to 0}\frac{\cos x - 1}{x} =$

36. If $f(a) = f(b) = 0$ and $f(x)$ is continuous on $[a,b]$, then
   a) $f(x)$ must be identically zero
   b) $f'(x)$ may be different from zero for all x on $[a,b]$
   c) there exists at least one number c, $a < c < b$, such that $f'(c) = 0$
   d) $f'(x)$ must exist for every x on (a,b)
   e) none of the preceding is true



37. Suppose $\lim_{x \to 0} \frac{g(x) - g(0)}{x} = 1$. It follows necessarily that

a) g is not defined at $x = 0$
b) g is not continuous at $x = 0$
c) The limit of $g(x)$ as x approaches 0 equals 1.
d) $g'(0) = 1$
e) $g'(1) = 0$

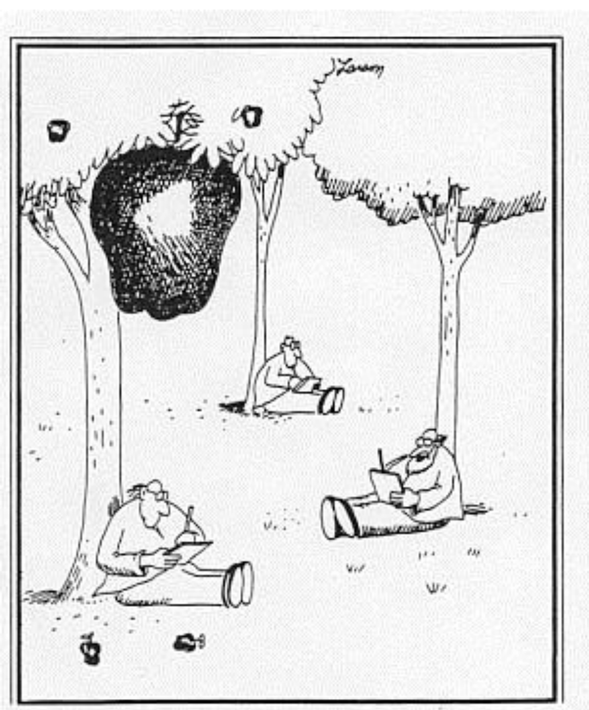
38. Find the instantaneous rate of change of the area of a circle with respect to a) its radius, r
b) its circumference, C

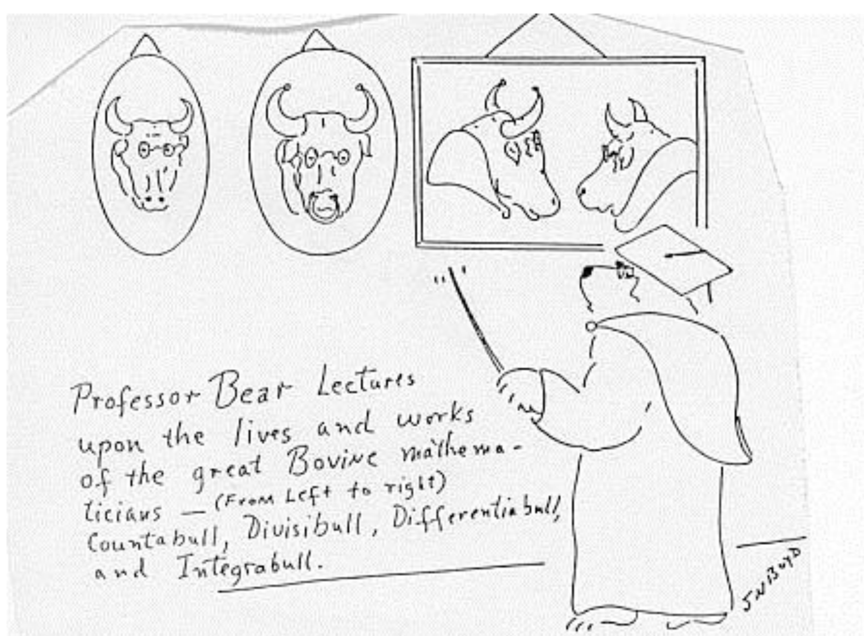
39. A particle moves along a straight line with position function $s(t) = 3t^3 - 11t^2 + 8t$. In what interval of t is the particle moving to the left along the line?

40. A ball is thrown directly upward with an initial velocity of 24.5 m/s and its height, h, in metres, is given by $h(t) = 24.5t - 4.9t^2$, where t is in seconds.

a) Find the velocity after 1 s.
b) When does it reach its maximum height?
c) What is its maximum height?
d) When does it strike the ground?
e) What is the acceleration?

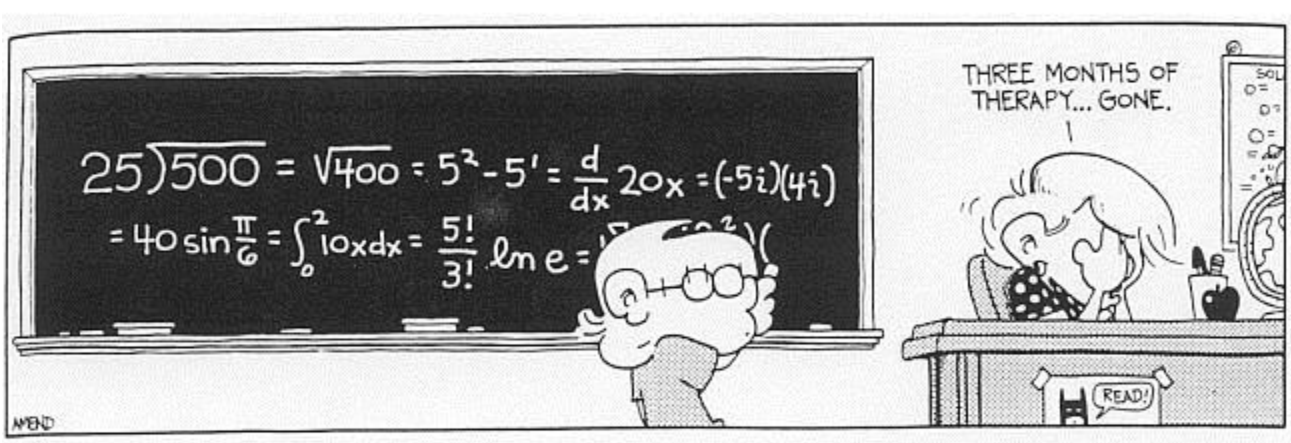
41. If the k^{th} derivative of $(3x-2)^4$ is zero, then what must be the case concerning k ?


"Nothing yet. ... How about you, Newton?"



Key

1. $(1-x)^2(1-16x)$	2. $-\dfrac{7}{(3x+1)^2}$	3. $-\dfrac{1}{\sqrt{3-2x}}$	4. $-30(5x+1)^{-4}$
5. $2x^{-\frac{1}{3}}-2x^{-\frac{1}{2}}$	6. $\dfrac{1}{\sqrt{x}}+\dfrac{1}{4x\sqrt{x}}$	7. $\dfrac{x+1}{\sqrt{x^2+2x+1}}$	8. $\dfrac{1}{(1-x^2)^{\frac{3}{2}}}$
9. $\dfrac{4x}{(1-x^2)^2}$	10. 2	11. -7	12. $\dfrac{1}{3}$
13. $\dfrac{2}{\sqrt{10}}$	14. $\dfrac{13}{25}$	15. -12	16. 6
17. 225	18. 4 only	19. 1, 2, and 6	20. $\dfrac{1}{\sqrt{3}}$
21. $\dfrac{y-3x^2}{3y^2-x}$	22. $\csc(x+y)-1$	23. $-\csc y\cos x$	24. $\dfrac{y-3x}{5y-x}$
25. $\{0,1,2\}$	26. $-\dfrac{1}{2}$	27. $-\dfrac{1}{5}$	28. $-c^2y$
29. 6	30. reals except $x=1$	31. 1	32. $-(x+1)^{-2}$
33. 6	34. $\dfrac{1}{12}$	35. 0	36. B
37. D	38a) $2\pi r$	b) $\dfrac{C}{2\pi}$	For those of you looking for secret messages, here it is: Make sure you ask Mr. O what it is.
39. $\dfrac{4}{9}<t<2$	40a) 14.7 *m/s*	b) 2.5 *s*	c) 30.6 *m*
d) after 5 s	e) $-9.8\ m/s^2$	41. $k\geq 5$	



Derivatives

Definition of the Derivative: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ $\qquad f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Derivative of a Constant

$$\frac{d}{dx} k = 0$$

Power Rule

$$\frac{d}{dx} x^n = n\, x^{n-1}$$

Constant times a Function

$$\frac{d}{dx} k \bullet f(x) = k \bullet f'(x)$$

Product Rule

$$\frac{d}{dx} f \bullet g = f'\, g + f\ g'$$

Quotient Rule

$$\frac{d}{dx} \frac{f}{g} = \frac{f'\, g - g'\, f}{g^2}$$

Chain Rule: $\dfrac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \bullet g'(x)$ or $\dfrac{dy}{dx} = \dfrac{dy}{du} \bullet \dfrac{du}{dx}$

Derivatives of Trigonometric Functions:

$$\frac{d}{dx} \sin u = \cos u \bullet \frac{du}{dx} \qquad \frac{d}{dx} \csc u = -\csc u \cot u \bullet \frac{du}{dx}$$

$$\frac{d}{dx} \cos u = -\sin u \bullet \frac{du}{dx} \qquad \frac{d}{dx} \sec u = \sec u \tan u \bullet \frac{du}{dx}$$

$$\frac{d}{dx} \tan u = \sec^2 u \bullet \frac{du}{dx} \qquad \frac{d}{dx} \cot u = -\csc^2 u \bullet \frac{du}{dx}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx} \sin^{-1} u = \frac{d}{dx} \arcsin u = \frac{1}{\sqrt{1 - u^2}} \bullet \frac{du}{dx} \quad |u| < 1 \qquad \frac{d}{dx} \cos^{-1} u = \frac{d}{dx} \arccos u = \frac{-1}{\sqrt{1 - u^2}} \bullet \frac{du}{dx} \quad |u| < 1$$

$$\frac{d}{dx} \tan^{-1} u = \frac{d}{dx} \arctan u = \frac{1}{1 + u^2} \bullet \frac{du}{dx} \qquad \frac{d}{dx} \cot^{-1} u = \frac{d}{dx} \operatorname{arc}\cot u = \frac{-1}{1 + u^2} \bullet \frac{du}{dx}$$

$$\frac{d}{dx} \sec^{-1} u = \frac{d}{dx} \operatorname{arc}\sec u = \frac{1}{|u| \bullet \sqrt{u^2 - 1}} \bullet \frac{du}{dx} \qquad \frac{d}{dx} \csc^{-1} u = \frac{d}{dx} \operatorname{arc}\csc u = \frac{-1}{|u| \bullet \sqrt{u^2 - 1}} \bullet \frac{du}{dx}$$

Exponential Functions: $\dfrac{d}{dx} e^u = e^u \dfrac{du}{dx} \qquad \dfrac{d}{dx} a^u = a^u \ln a \ \dfrac{du}{dx}$

Logarithmic Functions: $\dfrac{d}{dx} \ln u = \dfrac{1}{u} \dfrac{du}{dx} \qquad \dfrac{d}{dx} \log_b u = \dfrac{1}{u \ln b} \dfrac{du}{dx}$