if $f(x)$ and $g(x)$ are inverses $\qquad g'(b) = \frac{1}{f'(a)}$

3.7 Warmup $\quad (a,b)$ is on $f(x)$ $\quad (b,a)$ is on $g(x)$

1. If $f(x) = 2^x + x$ and $g(x) = f^{-1}(x)$, then what is $g'(3)$?

$f'(x) = 2^x \cdot \ln 2 + 1$

$g(x)$ has $(3, \underline{1})$

$g'(3) = \frac{1}{f'(1)}$

$f'(1) = 2 \cdot \ln 2 + 1$

$f(x)$ has $(\underline{1}, 3)$

$x = 2^y + y$

$g'(3) = \frac{1}{2\ln 2 + 1}$

2. If f and g are inverses of one another, with $f(0) = 2$, $f'(0) = -7$, $f(1) = 0$, and $f'(1) = 13$. What is $g'(0)$?

$g'(0) = \frac{1}{f'(1)}$

$f(x)$	$g(x)$
$(0,2)$	$(2,0)$
$\underline{(1,0)}$	$(0,1)$

$= \frac{1}{13}$

3. Determine the following derivatives:

a) $\frac{d}{dx} \log_3 \frac{2x^3}{x-4}$

$y' = 0 + 3\frac{1}{x\ln 3} - \frac{1}{(x-4)\ln 3}(1)$

$y = \log_3 2x^3 - \log_3 (x-4)$

eg $\quad y = \log x^4$

$y = \log_3 2 + 3\log_3 x - \log_3 (x-4)$

$= 4\log x$

b) $\frac{d}{dx} \ln 5^{x^2 + \sin x}$

$\boxed{y' = \ln 5 (2x + \cos x)}$

$y = (x^2 + \sin x)\ln 5$

4. Find the tangent(s) to $y = \ln x$ which have a slope of 2.

$y' = 2 \qquad y' = \frac{1}{x}$

$\boxed{y - \ln\frac{1}{2} = 2(x - \frac{1}{2})}$

$\frac{1}{x} = 2$

$y + \ln 2 = 2x - 1$

$x = \frac{1}{2} \qquad y = \ln(\frac{1}{2})$

Logarithmic Differentiation

Exponential Functions
(variable in the exponent)

$$y = a^x$$

$$y' = a^x \bullet \ln x$$

Power Functions
(variable in the base)

$$y = x^a$$

$$y' = a\, x^{a-1}$$

What about functions of the form $y = x^x$? This function is neither an exponential nor a power function, but seems to be some combination of the two.

One way to differentiate functions of this form is to use the fact that $x = e^{\ln x}$ $(x > 0)$ and to rewrite the base as this.

$$y = \left(e^{\ln x}\right)^x$$

$$y = e^{x \ln x}$$

$$y' = e^{x \ln x} \cdot \left[1 \cdot \ln x + \frac{1}{x} \cdot x\right]$$

$$y' = e^{x \cdot \ln x} \cdot [\ln x + 1]$$

$$y' = x^x \cdot (\ln x + 1)$$

Another way to differentiate functions of this form is to use a technique called **logarithmic differentiation.** This process involves the following:

a) take the logarithm of both sides of the equation
b) use logarithmic rules to rewrite the complicated side in the simplest form for differentiation
c) differentiate implicitly, solve for y' and then substitute for y

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} \cdot y' = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{1}{y} \cdot y' = \ln x + 1$$

$$y' = y(\ln x + 1)$$

$$\boxed{y' = x^x(\ln x + 1)}$$

This technique can also be used to differentiate complicated functions involving products, quotients or radicals of functions.

Determine the derivative for each of the following:

1) $y=\dfrac{x^3\sqrt{16-5x}}{(3x-7)^2}$

$$\ln y = \ln\left(\frac{x^3\sqrt{16-5x}}{(3x-7)^2}\right)$$

$$\ln y = \ln x^3 + \ln\sqrt{16-5x} - \ln(3x-7)^2$$

$$\ln y = 3\ln x + \tfrac{1}{2}\ln(16-5x) - 2\ln(3x-7)$$

$$\frac{1}{y}\cdot y' = 3\cdot\frac{1}{x} + \frac{1}{2}\left(\frac{1}{16-5x}\right)\cdot(-5) - 2\left(\frac{1}{3x-7}\right)\cdot(3)$$

$$y' = \left(\frac{x^3\sqrt{16-5x}}{(3x-7)^2}\right)\left[\frac{3}{x} - \frac{5}{2(16-5x)} - \frac{6}{3x-7}\right]$$

2) $y=\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

$$\ln y = \ln\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{\frac{1}{2}}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{1-\cos\theta}{1+\cos\theta}\right)$$

$$\ln y = \frac{1}{2}\left[\ln(1-\cos\theta) - \ln(1+\cos\theta)\right]$$

$$\frac{1}{y}\cdot y' = \frac{1}{2}\left[\frac{1}{1-\cos\theta}\cdot\sin\theta - \frac{1}{1+\cos\theta}\cdot -\sin\theta\right]$$

$$y' = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(\frac{1}{2}\right)\left(\frac{\sin\theta}{1-\cos\theta} + \frac{\sin\theta}{1+\cos\theta}\right)$$

$$y' = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(\frac{1}{2}\right)\left[\frac{\sin\theta+\sin\theta\cos\theta}{1-\cos^2\theta} + \frac{\sin\theta-\sin\theta\cos\theta}{1-\cos^2\theta}\right]$$

$$y' = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(\frac{1}{2}\right)\left(\frac{2\sin\theta}{\sin^2\theta}\right)$$

$$y' = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\cdot\frac{1}{\sin\theta}$$

3) $y=(\sin x)^x$

$$\ln y = x\ln(\sin x)$$

$$\frac{1}{y}\cdot y' = x\cdot\frac{1}{\sin x}\cdot\cos x + \ln\sin x$$

$$y' = (\sin x)^x\cdot\left[x\cdot\frac{\cos x}{\sin x} + \ln\sin x\right]$$

$$y' = (\sin x)^x\cdot\left[x\cdot\cot x + \ln\sin x\right]$$

4) $y = x^{\sin x}$

$\ln y = \sin x \cdot \ln x$

$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$

$y' = \left(x^{\sin x}\right)\left(\cos x \ln x + \frac{\sin x}{x}\right)$

5) $y = \ln x + \ln y$

implicit differentiation

$y' = \frac{1}{x} + \frac{1}{y} \cdot y'$

$y' - \frac{1}{y} \cdot y' = \frac{1}{x}$

$y'\left(1 - \frac{1}{y}\right) = \frac{1}{x}$

$y' = \frac{\frac{1}{x}}{\left(1 - \frac{1}{y}\right)}$

$y' = \frac{1}{x} \cdot \frac{1}{\left(1 - \frac{1}{y}\right)}$

$= \frac{1}{x} \cdot \frac{1}{\frac{y-1}{y}} = \frac{y}{x(y-1)}$

6) $y = \ln \sin^2 x$

$y = 2 \ln \sin x$

$y' = 2 \frac{1}{\sin x} \cdot \cos x$

$y' = \quad \cdot 2 \cot x$

7) $\ln xy = 3 - 2x + 5y$

$\frac{1}{xy} \cdot (y + xy') = 0 - 2 + 5y'$

$\frac{1}{x} + \frac{1}{y} y' = -2 + 5y'$

$\frac{1}{y} y' - 5y' = -2 - \frac{1}{x}$

$y'\left(\frac{1}{y} - 5\right) = \frac{\left(-2 - \frac{1}{x}\right)}{\left(\frac{1}{y} - 5\right)} = \frac{\left(\frac{-2x-1}{x}\right)}{\left(\frac{1-5y}{y}\right)} = \frac{(-2x-1)(y)}{x(1-5y)}$