Separable Differential Equations (both x and y in $\frac{dy}{dx}$)

A **differential equation** is any equation that involves a unknown function and its derivative(s) or differential(s). To solve a differential equation means to find the function or set of functions that satisfies the equation. - it is the result of implicit differentiation

A differential equation is separable if it can be expressed in the form $f(y)dy = f(x)dx$. In other words, the variables can be separated with their corresponding differential. Once the equation is expressed in this way, one takes the integral of both sides.

Example 1. Solve the differential equation: $\frac{dy}{dx} = \frac{-4x}{3y}$

a) Find the general solution
b) Identify the family of curves defined by the solution
c) Find the particular solution if $f(-1) = 2$

a)

$$3y \cdot dy = -4x \cdot dx$$

$$\int 3y \, dy = \int -4x \, dx$$

$$\frac{3}{2}y^2 = \frac{-4}{2}x^2 + C$$

$$\boxed{\frac{3}{2}y^2 = -2x^2 + C}$$

c)

$$\frac{3}{2}(2)^2 = -2(-1)^2 + C$$

$$6 = -2 + C$$

$$C = 8$$

$$\boxed{\frac{3}{2}y^2 = -2x^2 + 8}$$

Example 2. a) Find the general solution to $\frac{dy}{dx} = 4xy^2$

b) Find the particular solution if $(1,-1)$ lies on the graph of $\underline{y = f(x)}$

a)

$$\frac{dy}{dx} = 4xy^2$$

$$\int y^{-2} \, dy = \int 4x \, dx$$

$$-1y^{-1} = 2x^2 + C$$

$$\boxed{-\frac{1}{y} = 2x^2 + C}$$

b)

$$\frac{-1}{(-1)} = 2(1)^2 + C$$

$$1 = 2 + C$$

$$C = -1$$

$$-\frac{1}{y} = 2x^2 - 1$$

$$-\frac{y}{1} = \frac{1}{2x^2 - 1}$$

$$\boxed{y = \frac{-1}{2x^2 - 1}}$$

Example 3. Solve: $\frac{dy}{dx} = xy - 5y$

$$\frac{dy}{dx} = y(x-5)$$

$$\int \frac{1}{y}\,dy = \int (x-5)\,dx$$

$$\ln y = \frac{1}{2}x^2 - 5x + c$$

$$e^{\ln y} = e^{(\frac{1}{2}x^2 - 5x + c)}$$

$$y = e^{\frac{1}{2}x^2 - 5x + c}$$

$$y = \frac{e^{\frac{1}{2}x^2}}{e^{5x}} \cdot e^{c}$$

$$\boxed{y = c \cdot \frac{e^{\frac{1}{2}x^2}}{e^{5x}}}$$

Example 4. Solve: $\frac{dy}{dx} = \frac{x\sin^2 y^2}{y}$

$$\int \frac{y}{\sin^2(y^2)}\,dy = \int x\,dx$$

$$u = y^2$$

$$\frac{du}{dy} = 2y$$

$$dy = \frac{1}{2y} \cdot du$$

$$\int \frac{\cancel{y}}{\sin^2(u)} \cdot \frac{1}{2\cancel{y}}\,du = \frac{1}{2}x^2 + c$$

$$\frac{1}{2}\int \frac{1}{\sin^2(u)}\,du = \frac{1}{2}x^2 + c$$

$$\frac{1}{2}\int \csc^2 u\,du = \frac{1}{2}x^2 + c$$

$$-\frac{1}{2}\cot u = \frac{1}{2}x^2 + c$$

$$\boxed{-\frac{1}{2}\cot(y^2) = \frac{1}{2}x^2 + c}$$

Example 5. Suppose that the function y satisfies the differential equation $\frac{dy}{dt} = -8y$ and let $\boxed{z = y^2}$. Then z satisfies a differential equation of the form $\frac{dz}{dt} = f(z)$. Find $f(z)$. we want $\frac{dz}{dt}$

$$\frac{dz}{dt} = \frac{dy}{dt} \cdot \frac{dz}{dy}$$

$$z = y^2$$

$$\frac{dz}{dy} = 2y$$

$$\frac{dz}{dt} = -8y \cdot 2y$$

$$\frac{dz}{dt} = -16y^2 \quad \text{since } y^2 = z$$

$$\frac{dz}{dt} = -16z$$

$$\boxed{f(z) = -16z}$$