Related Rate Problems

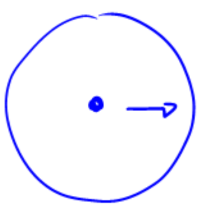
If a variable y is a function of time t, then the rate of change of y with respect to t is given by $\frac{dy}{dt}$. If several variables are functions of time t and can be related by an equation, we can obtain a relation involving their rates of change by finding derivatives with respect to t by applying the chain rule.

A ***related rate problem*** is a problem that presents a situation whereby one or more related quantities are changing and that asks for the rate at which one of the quantities is changing. The key step in solving such a problem involves writing an equation that relates the variable quantities in the problem. The steps involved in solving such a problem may be outlined as follows:

1. Read the problem carefully, ensuring that you fully understand it.
2. Draw a diagram if possible.
3. Introduce appropriate notation. Assign symbols to all quantities that are functions of time.
4. Express the **given** information and the **required** rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary use the geometric concepts involved in the situation to eliminate one of the variables by substitution.
6. Use the chain rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Examples

1. An oil spill from a tanker spreads out in a circular pattern, centered at the tanker's position. If the perimeter if the spill is moving outwards at 2 m/s, find the rate of increase of the contaminated area when the radius is 500 m.



$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dr}{dt} = 2$

$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

$= (2\pi r)(2) \Big|_{r=500}$

$= 2\pi(500)(2)$

$\frac{dA}{dt} = 2000\pi \text{ m}^2/\text{s}$

2. A spherical snowball is melting in such a way that its volume is decreasing at a rate of $1\,\text{cm}^3/\text{min}$. At what rate is the radius decreasing when the radius is 5 cm?

$\frac{dV}{dt} = -1$

want $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$

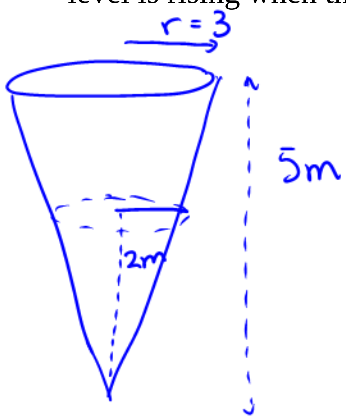
$V = \frac{4}{3}\pi r^3$

$= \frac{1}{4\pi r^2} \cdot -1 \Big|_{r=5}$

$\frac{dV}{dr} = 4\pi r^3$

$= \frac{-1}{4\pi(5)^2} = \frac{-1}{100\pi}$ cm/min

3. A water tank is built in the shape of a circular cone with a height of 5 m and a diameter of 6 m at the top. Water is being poured into the tank at a rate of $1.6\ \text{m}^3/\text{min}$. Find the rate at which the water level is rising when the water is 2 m deep.



$\frac{dV}{dt} = 1.6$

$V = \frac{1}{3}\pi r^2 \cdot h$

$\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt}$

$* \frac{3}{5} = \frac{r}{h}$

$h = \frac{5}{3} r$

$\frac{dh}{dr} = \frac{5}{3}$

$V = \frac{1}{3}\pi r^2 \left(\frac{5r}{3}\right)$

$V = \frac{5\pi}{9} \cdot r^3$

$\frac{dV}{dr} = \frac{5\pi}{3} \cdot r^2$

$= \frac{5}{3}\left(\frac{3}{5\pi r^2}\right) \cdot 1.6$

$\frac{dh}{dt} = \frac{1.6}{\pi r^2}$

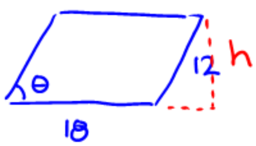
$= \frac{1.6}{\pi (1.2)^2}$

$= 0.354\ m/min$

$\boxed{\frac{3}{5} = \frac{r}{2} \quad r = 1.2}$

$\boxed{\text{p237 \#1-15}}$

4. A parallelogram with sides of 12 cm and 18 cm is hinged at each corner. If the angle between the two sides is increasing at 2^0 per minute, at what rate is the area of the parallelogram increasing when the angles between the sides is 30^0?



$A = b \times h$

$\frac{dA}{dt}$

$\sin\theta = \frac{h}{12}$

$h = 12\sin\theta$

$A = 18 \cdot 12 \sin\theta$

$\frac{dA}{d\theta} = 216\cos\theta$

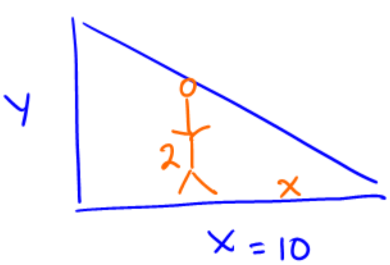
$\frac{d\theta}{dt} = 2^\circ \times \frac{\pi}{180}$

$\frac{d\theta}{dt} = \frac{\pi}{90}$

$\frac{dA}{dt} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dt}$

$= 216\cos\left(\frac{\pi}{6}\right) \cdot \frac{\pi}{90} = 216 \times \frac{\sqrt{3}}{2} \times \frac{\pi}{90} = \frac{216\pi\sqrt{3}}{180}\ \frac{cm^2}{min.}$

5. A spotlight on the ground shines on a wall 10 m away. A man 2 m tall walks from the spotlight toward the wall at a speed of $1.2\ \text{m}/\text{s}$. How fast is his shadow on the wall decreasing when he is 3 m from the wall?



$x = 10$

$\frac{dx}{dt} = 1.2$

$\frac{x}{2} = \frac{10}{y}$

$y = \frac{20}{x}$

$\frac{dy}{dx} = \frac{-20}{x^2}$

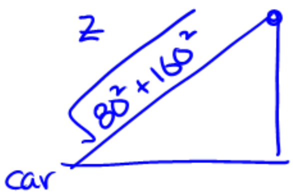
need $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$= \frac{-20}{x^2} \cdot 1.2 \Big|_{x=7}$

$= \frac{-20(1.2)}{7^2}$

$\frac{dy}{dt} = -.49\ m/s$

* 6. A car traveling at 80 km/h travels south for 1 hour and then heads west. How fast is the distance from the car to the starting point increasing when the car has been traveling for 3 hours? need $\frac{dz}{dt}$



$y = 80$

$\frac{dy}{dt} = 0$

at 3hours

$x = 160$

$\frac{dx}{dt} = 80$

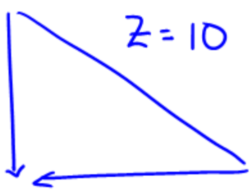
$z^2 = x^2 + y^2$

$2z \cdot \frac{dz}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$

$\frac{dz}{dt} = \frac{2(160)(80) + 2(80)(0)}{2\sqrt{80^2 + 160^2}}$

$= 71.55$ km/h

7. Two cars approach a right angled intersection, one traveling south at 50 km/h and the other west at 80 km/h. When the fastest car is 8 km from the intersection and the other car is 6 km from the intersection, how fast is the distance between the cars changing?



$y = 6$

$\frac{dy}{dt} = 50$

$x = 8$

$\frac{dx}{dt} = 80$

want $\frac{dz}{dt}$

$z^2 = x^2 + y^2$

$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$

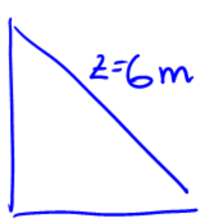
$\frac{dz}{dt} = \frac{2(8)(80) + 2(6)(50)}{2(10)}$

$\frac{dz}{dt} = 94$ km/h

8. A 6 m ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 1 m/sec. how fast will the top of the ladder be moving down the wall when it is 2 m above the ground?

$\frac{dy}{dt} = ?$

$z^2 = x^2 + y^2$



$y = 2$

$\frac{dz}{dt} = 0$

$x = ?$

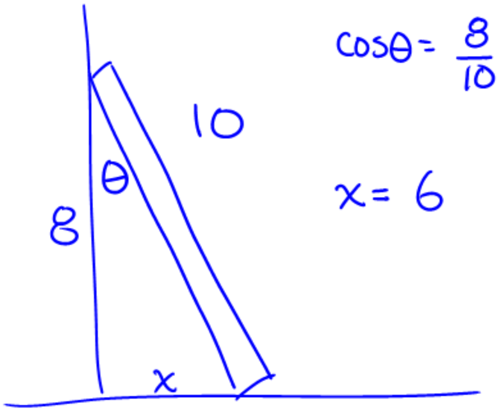
$\frac{dx}{dt} = 1$

$x^2 = \sqrt{6^2 - 2^2}$

$x = \sqrt{32}$

p237 #1-15.

9. A 10 m ladder is leaning against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a rate of 20 cm/sec, how fast is the angle between the ladder and the wall changing when the top of the ladder is 8 m above the ground?



$\rightarrow \frac{dx}{dt} = 0.2$

$\cos\theta = \frac{8}{10}$

$x = 6$

$\sin\theta = \frac{x}{10}$

need $\frac{d\theta}{dt} = \quad \cdot \frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{d\theta}{\cancel{dx}} \cdot \frac{\cancel{dx}}{dt}$

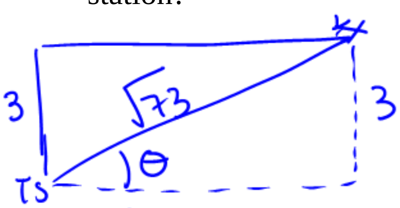
$\cos\theta \cdot \frac{d\theta}{dx} = \frac{1}{10}$

$\frac{d\theta}{dx} = \frac{1}{10\cos\theta}$

$= \frac{1}{10\left(\frac{8}{10}\right)} \cdot 0.2$

$= .025$ radians/second.

10. An aircraft is flying at an altitude of 3 km at a speed of 600 km/h in a direction away from a tracking station. How fast is the angle of elevation changing when the aircraft is over a point 8 km from the station?



8 $\quad \frac{dx}{dt} = 600$

need $\frac{d\theta}{dt} = \frac{d\theta}{dx} \quad \frac{dx}{dt}$

$\tan\theta = \frac{3}{x}$

$\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{-3}{x^2}$

$\frac{d\theta}{dx} = \frac{-3}{\sec^2\theta \cdot x^2}$

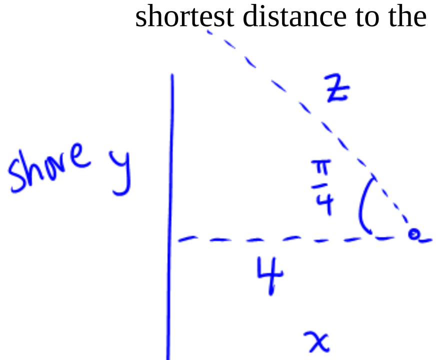
$\frac{d\theta}{dx} = \frac{-3\cos^2\theta}{x^2}$

$= \frac{-3\cos^2\theta}{x^2} \cdot 600$

$= \frac{-3\left(\frac{8}{\sqrt{73}}\right)^2}{(8)^2} \cdot 600$

$= -24.66$ radians/hour

11. A radar antenna is located on a ship that is 4 km from a straight shore. It is rotating at 32 rev/min. How fast does the radar beam sweep along the shore when the angle between the beam and the shortest distance to the shore is $\frac{\pi}{4}$?



$\frac{d\theta}{dt} = 32 \frac{rev}{min} \times \frac{2\pi^R}{rev}$

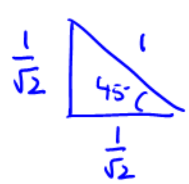
$\frac{d\theta}{dt} = 64\pi$ /min

want $\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}$

$\frac{y}{4} = \tan\theta$

$\frac{1}{4} = \sec^2\theta \cdot \frac{d\theta}{dy}$

$\frac{dy}{d\theta} = 4\sec^2\theta$



$= 4\sec^2\theta \cdot 64\pi$

$= 4\sec^2\left(\frac{\pi}{4}\right) \cdot 64\pi$

$= 4\left(\sqrt{2}\right)^2 \cdot 64\pi$

$\boxed{\frac{dy}{dt} = 512\pi \text{ rad/min}}$

$\boxed{\begin{array}{l} \text{P237-240} \\ \#1-15, 17-25 \\ 27, 29-37 \end{array}}$