Differentials

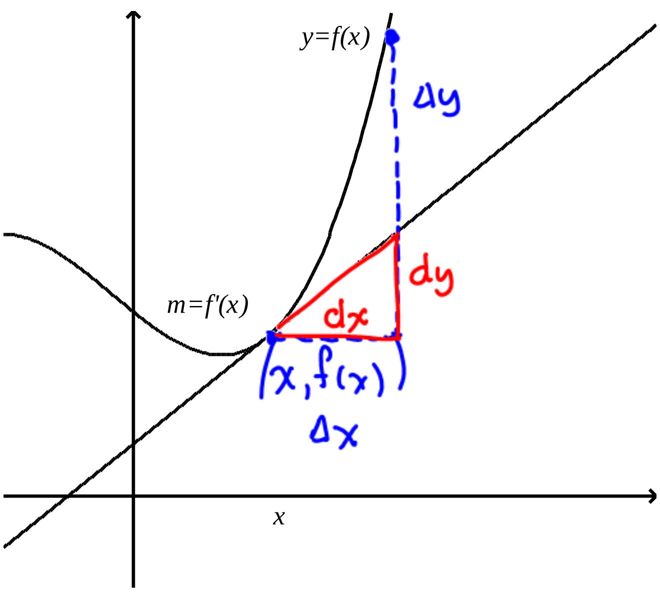
Estimating Change in a Function

In the definition of the derivative, namely:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

$\frac{dy}{dx}$ is an operator, not a fraction. It is possible, however, to assign meanings to dy and dx independently. Leibnitz tried to define them as very small quantities greater than zero but less than any positive real number. He referred to them as *infinitesimals*. That is $\frac{dy}{dx} = \frac{\textit{very small } \Delta y}{\textit{very small } \Delta x}$

Although he used this idea with some success, it was ultimately the source of much confusion. Instead, we can **define** $dx = \Delta x$. And although dy and Δy are not exactly the same, for very small dx (or Δx), dy is very close to Δy. In other words, as $\Delta x \to 0$, $dy \to \Delta y$. The diagram below helps explain this idea:



$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \quad \text{for "very small" } \Delta x$$

dy and dx are variables

dx is Δx if Δx is very small

dy is dependent on x & Δx

$\frac{dy}{dx} = f'(x)$

$dy = f'(x) \cdot dx$

We call dy the **differential** of the function $y = f(x)$ and this is defined as :

$$dy = f'(x)dx$$

The differential allows us to obtain approximations to Δy for small Δx (which by definition is the same as dx)

1. Compare Δy and dy for $y = x^2$ as x changes from $\overset{x}{2}$ to 2.01

$\Delta y = y(2.01) - y(2)$

$= (2.01)^2 - (2)^2$

$\Delta y = .0401$

= absolute change in Δy from $x=2 \to x=2.01$

$dy = f'(x) \cdot dx$ "estimate" using derivative

$= 2(2) \cdot (2.01 - 2)$

$dy = 0.04$

2. Using differentials, find the approximate increase in the volume of a soap bubble if its radius increases from 3 cm to 3.025 cm. Compare this approximation to the actual ΔV.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$dV = 4\pi r^2 \cdot dr$

$= 4\pi (3)^2 (3.025 - 3)$

$dV = 2.827$

$\Delta V = V(3.025) - V(3)$

$= \frac{4}{3}\pi (3.025)^3 - \frac{4}{3}\pi (3)^3$

$\Delta V = 2.8511$

Describing Change

As the variable x changes from a to $a + dx$, the change in y (the function) can be described in three ways

	Actual	Estimated
Absolute change - amount it changes by	$\Delta f = f(a+dx) - f(a)$	$df = f'(a)dx$
Relative change - amount compared to y.	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$

Determine the actual and estimated percentage change for question #2 above.

differentials $dV = 2.827$ absolute change.

$\frac{dV}{V(3)} = \frac{2.827}{113.0973} = 0.024996$ relative change

$= 2.5\%$ % change

actual $\Delta V = 2.8511$

$\frac{\Delta V}{V(3)} = 0.02521$

$= 2.5\%$ change

3. A manufacturer contracts to mint coins for the government. How much variation dr in the radius of the coins can be tolerated if the coins are to weight within .1% of their ideal weight?

weight is dependent on volume.

$\frac{dV}{V} = 0.1\%$

$dV = V' \cdot dr$

$\frac{dV}{V} = .001$

$\frac{V' \cdot dr}{V} = .001$

cylinder

$V = \pi r^2 \cdot h$

$\frac{dV}{dr} = 2\pi r \cdot h$

$\frac{dV}{V} = \frac{2\cancel{\pi} r \cdot \cancel{h} \cdot dr}{\cancel{\pi} r^2 \cdot \cancel{h}} = .001$

$\frac{2 \cdot dr}{r} = .001$

$dr = \frac{.001 \cdot r}{2}$

$dr = \frac{1}{2000} r$

tolerating a difference of $\frac{1}{2000}$ of radius or .05% of radius

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