# *Optimization Problems (Applied Max and Min)*

Many problems in science, engineering, economics and geometry involve the determination of the maximum or minimum value of some varying quantity. These are called optimization problems. A suggested approach for the solution of such a problem is a follows:

1. Read the problem carefully, several times if necessary, and decide what quantity is to be maximized or minimized. After assigning a letter to that quantity, express it as a function of the other variable or variables in the problem.

2. Draw a diagram if possible.

3. Often more than one equation will be involved. However, ultimately one must express the quantity to be optimized as a function of a single variable. This may require some substitutions from secondary equations.

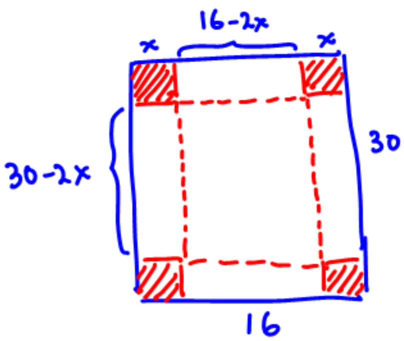
4. Find the derivative of the function from step 3 and determine the critical points. Determine which are maxima and/or minima. Don't forget to check the endpoints of the domain. Check to see if your answer is reasonable.

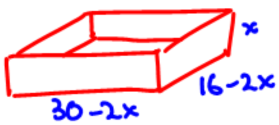
Examples

1. What is the area of the largest rectangle with base on the $x$-axis and one vertex at the origin that can fit inside the triangle whose sides are bounded by the coordinate axes and $5x + 2y = 10$?

2. The sum of two non-negative numbers is $A$. Find the numbers if their product is as small as possible; as large as possible.

3. An open box is to be made from a 16 dm by 30 dm piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be in order to construct a box with the maximum volume?




$V = l \cdot w \cdot h$

$V = (x)(16-2x)(30-2x)$

$= x(480 - 92x + 4x^2)$

$V = 4x^3 - 92x^2 + 480x$

$V' = 12x^2 - 184x + 480$

critical points when $V' = 0$

$0 = 12x^2 - 184x + 480$

$x = \dfrac{-(-184) \pm \sqrt{(-184)^2 - 4(12)(480)}}{2(12)}$

$x = \cancel{12}$ or $3.\overline{3}$

$x = 3.33$ is critical point

$y''$ helps us determine if conc up, conc down or inflection point.

$y'' = 24x - 184 \Big|_{x = 3.\overline{3}}$

$y'' = -104$ $\therefore$ concave down

$x = 3.33$ must make a max volume

4. A closed cylindrical container is to hold 1 litre of liquid. How should the height and radius be chosen in order to minimize the amount of material used to construct the can?

$1 \text{ L} = 1000 \text{ cm}^3$

$V = 1000 = \pi r^2 \cdot h \longrightarrow h = \dfrac{1000}{\pi r^2}$

minimize SA

$SA = 2\pi r^2 + 2\pi r h$

$SA = 2\pi r^2 + 2\cancel{\pi} r \cdot \dfrac{1000}{\cancel{\pi} r^2}$

$SA = 2\pi r^2 + \dfrac{2000}{r}$

$SA = 2\pi r^2 + 2000 \cdot r^{-1}$

$SA' = 4\pi r + (-1)(2000) r^{-2}$

$0 = 4\pi r - \dfrac{2000}{r^2}$

$\dfrac{2000}{r^2} = 4\pi r$

$2000 = 4\pi r^3$

$r^3 = \dfrac{2000}{4\pi}$

$r = \sqrt[3]{\dfrac{2000}{4\pi}}$

$r = 5.42$

$SA'' = 4\pi + (-1)(-2)(2000) r^{-3}$

$SA'' = 4\pi + \dfrac{4000}{r^3}$

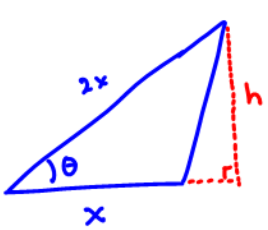
when $5.42$ $SA'' > 0$

$\therefore$ conc up

minimum SA when $r = 5.42$

$h = 10.84 \longleftarrow h = \dfrac{1000}{\pi (5.42)^2}$

5. In a triangle, one side is twice as long as the other side. What should the angle between these two sides be to maximize the area of the triangle? What is the maximum area?




$\sin\theta = \dfrac{h}{2x}$

$h = 2x \cdot \sin\theta$

$A = \frac{1}{2} \times b \times h$

$= \frac{1}{2}(x)(2x \sin\theta)$

$A = x^2 \cdot \sin\theta$

$\dfrac{dA}{d\theta} = x^2 \cdot \cos\theta$

$0 = x^2 \cdot \cos\theta$

$0 = \cos\theta$ $\quad \boxed{\theta = \frac{\pi}{2}}$

$\dfrac{d^2A}{d\theta^2} = -x^2 \sin\theta \Big|_{\frac{\pi}{2}}$

$= -x^2$

conc down

$\therefore$ max area at $\frac{\pi}{2}$

$A = \frac{1}{2}(x)(2x)(\sin\theta)$

$= \frac{1}{2}(x)(2x)(1)$

$\boxed{A = x^2}$

when $\theta = \frac{\pi}{2}$

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6. Find a point on the curve $y = x^2$ closest to $(18, 0)$.

7. At one o'clock ship A, sailing due east at 25 km/h is 100 km due north of ship B, which is sailing due north at 40 km/h. At what time are the ships nearest to each other and what is their minimum distance apart?

8. Grant's farm is 2 km away from the highway and from the highway, it is then 10 km into town. Grant's cows are hungry, and he needs to get into town and return as quickly as possible to obtain some feed. If he can travel at 80 km/h over the open field and 100 km/h on the highway, at what point on the highway should he aim for to get to town as quickly as possible?

9. A sphere has a radius of $r$. A right cylinder with radius $R$ and height $H$ is placed inside the sphere. What values of $R$ and $H$ should be chosen to maximize the volume of the cylinder?