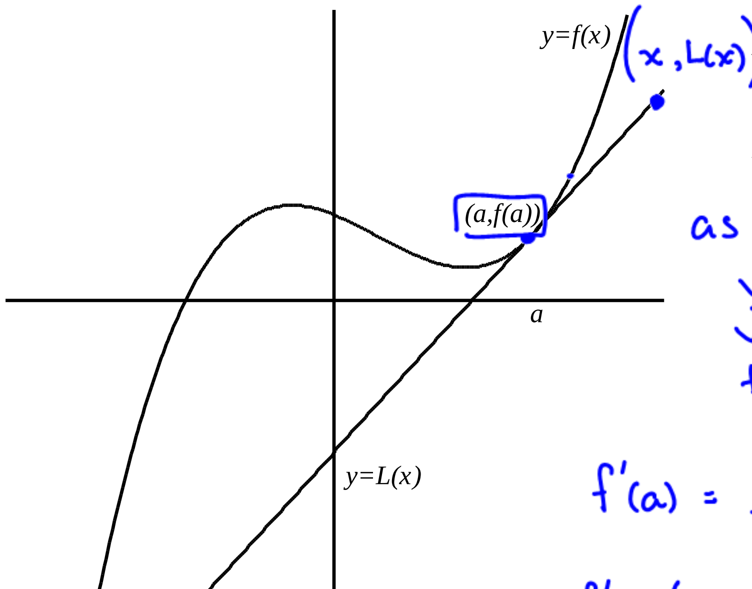
Linear Approximation

How does your calculator determine values like $\sqrt{.5}$, $\sin .5$, $\log .5$, $e^{.5}$ etc. ? There is no "magic table" that your calculator access to come up with these values (your calculator still has only a finite memory). Most of these "complicated" functions have their values determined by using a "simpler" function to estimate to a very high degree of accuracy the value of the function. ***Linear approximation*** is simply using a ***linear function*** as an approximating function for one that is more complicated.

Visually



Note: For values "close" to a, the value of the function, $f(x)$, is reasonably close to the value of the tangent, $L(x)$.

as you zoom way in y-value of tangent line is very close to the y-value of the function.

$$f'(a) = \frac{L(x) - f(a)}{x - a}$$

$$f'(a)(x-a) = L(x) - f(a)$$

$$\boxed{L(x) = f'(a)(x-a) + f(a)}$$

Thus the Linear Approximation or Tangent Line Approximation of f at $x = a$ is

$$L(x) = f'(a)(x-a) + f(a)$$

This function is called the ***local linearization*** or simply the ***linearization*** of f at $x = a$, and means that

$$f(x) \approx f'(a)(x-a) + f(a)$$ provided x is very close to a.

Other polynomial approximating functions can be found which provide a greater accuracy for larger intervals around x. A ***quadratic approximation*** is $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

equation using tangent

* 1. a) Find the linearization of the function $f(x)=\sqrt{x+1}$ at $x=3$ and use it to approximate $\sqrt{3.97}$ and $\sqrt{4.06}$.

b) Will the approximations be greater than or less than the actual values? checking concavity.

c) How close are the approximations to the actual values? (ie. How much error is there in the approximation)

$f(x)=\sqrt{x+1}$

$f'(x)=\frac{1}{2}(x+1)^{-\frac{1}{2}}$

$=\frac{1}{2\sqrt{x+1}}$

$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)(x+1)^{-3/2}$

$=\frac{-1}{4\sqrt{x+1}^{\,3}}$

near $x=3$

$L(x)=f'(a)(x-a)+f(a)\,\Big|_{a=3}$

$=\frac{1}{4}(x-3)+2$

$L(x)=\frac{1}{4}x-\frac{3}{4}+2$

$L(x)=\frac{1}{4}x+\frac{5}{4}$ when x is close to 3

$f(x)=\sqrt{3.97}=\sqrt{x+1}$

use $x=2.97$

$L(2.97)=\frac{1}{4}(2.97)+\frac{5}{4}$

$=1.9925$

$\sqrt{3.97}=1.99248585$

concave down

Linear is above the function, it is too big.

2. Find the linear approximation to $y=\sin x$ at $i)$ $x=0$ and at $\cancel{ii)\ x=\frac{\pi}{2}}$

$L(x)=f'(a)(x-a)+f(a)$

$f(x)=\sin x\,\Big|_{x=0}$

$=0$

$f'(x)=\cos x\,\Big|_{x=0}$

$=1$

$L(x)=(1)(x-0)+0$

$L(x)=x$ provided that x is close to 0

3. The following facts are known about the function $f(x)$. (2003 Challenge Exam)

$i)$ $f(2)=4$ $\qquad ii)$ $f'(x)=\left(x^4+1\right)^{-1}$ for all x

a) Use linear approximation to estimate $f(2.05)$. Call your answer α

b) Circle the correct statement: $\alpha<f(2.05)$ $\qquad \alpha=f(2.05)$ $\qquad \alpha>f(2.05)$

Justify your answer without finding an antiderivative of $\left(x^4+1\right)^{-1}$