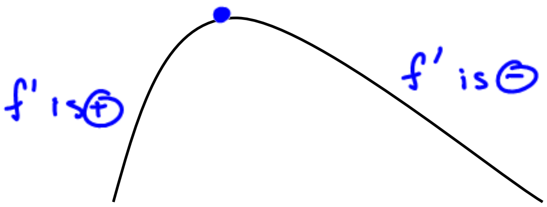
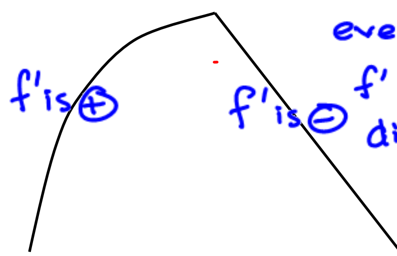
Maximum , Minimum and Inflection Points

First Derivative Test

For a continuous function f with critical point c f(c) does exist

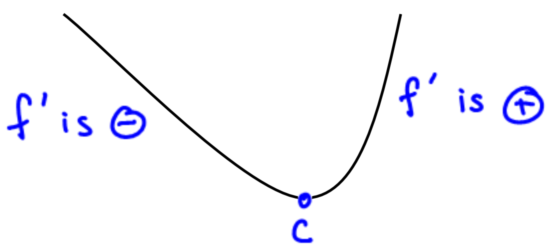
1) f has a local maximum at c if f' changes sign from __+__ to __−__ at c

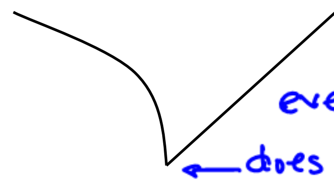




even if f' is not differentiable at c

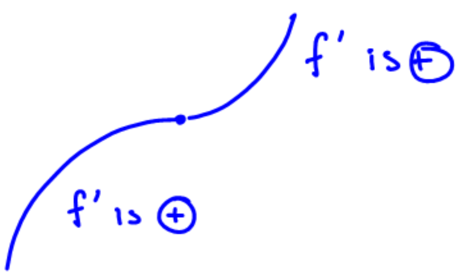
2) f has a local minimum at c if f' changes sign from __−__ to __+__ at c





even if f' does not exist

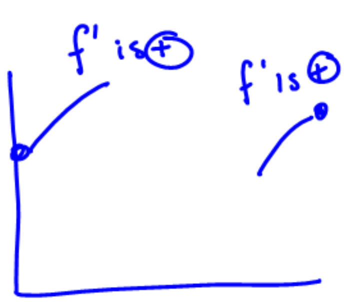
3) f has a no local extreme at c if f' does not change sign at c



if there is no sign change for f' then there are no local extrema

At the endpoints of an interval

look at the behaviour just to the left/right of endpoint



Examples

1. Find the critical points of $f(x) = 2x^3 - 6x^2 + 1$. Determine the functions local and absolute extreme values. Indicate intervals on which f is increasing and decreasing.

a) find critical points.

$f'(x) = 6x^2 - 12x$

$0 = 6x^2 - 12x$

$= 6x(x-2)$

$x = 0, 2$

$-1 \quad 1 \quad 3$

$0 \quad 2$

$y' \quad 18 \quad 0 \quad -6 \quad 0 \quad 18$

local max

local min

local max $= (0, f(0))$ or $(0, 1)$

local min $= (2, f(2))$ or $(2, -7)$

increasing $(-\infty, 0)$ and $(2, \infty)$

decreasing $(0, 2)$

2. Find the critical points of $f(x) = (x^2 - 3)e^x$. Determine the functions local and absolute extreme values. Indicate intervals on which f is increasing and decreasing.

$f'(x) = (2x)e^x + e^x(x^2-3)$

$f'(x) = e^x(2x + x^2 - 3)$

$0 = e^x(x+3)(x-1)$

$x = -3, 1$

$-4 \quad 0 \quad 2$

$-3 \quad 1$

$y' \quad + \quad 0 \quad -3 \quad 0 \quad 5e^2$

$- \quad +$

local max at $(-3, f(-3)) = (-3, 6e^{-3})$

local min at $(1, f(1)) = (1, -2e)$

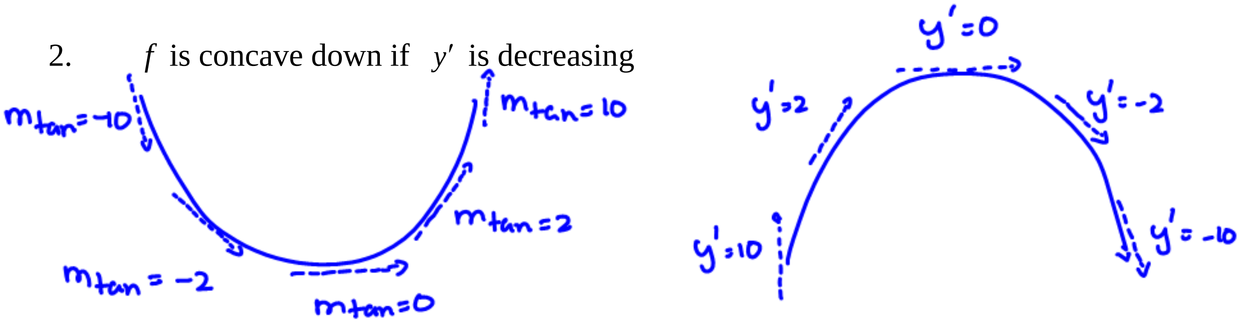
increasing on $(-\infty, -3)$ and $(1, \infty)$

decreasing on $(-3, 1)$

Definition of Concavity

Given $y = f(x)$ is a differentiable function, then

1. f is concave up if y' is increasing
2. f is concave down if y' is decreasing



concave up — y' is increasing

concave down — y' is decreasing

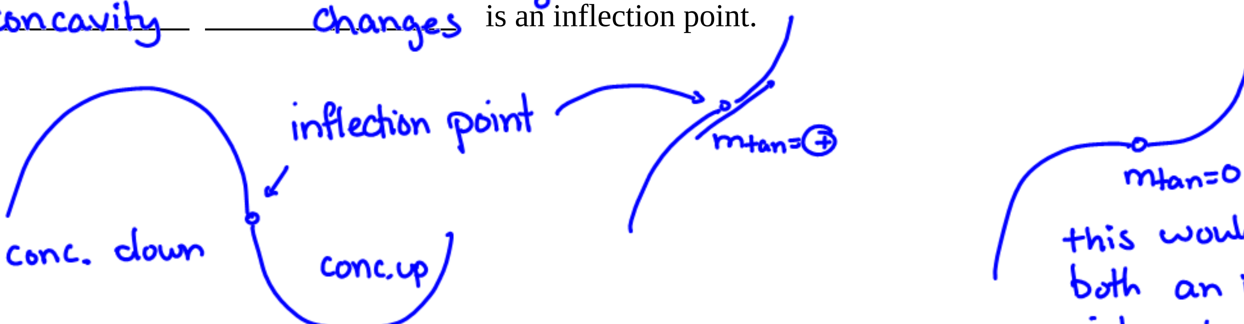
If $y = f(x)$ is a twice differentiable function, then $y = f(x)$ is

a) concave __up__ on any interval where $y'' > 0$

b) concave __down__ on any interval where $y'' < 0$

Inflection Points

Any point where the function has a __tangent__ __line__ and where __concavity__ __changes__ is an inflection point.



this would be both an inflection point and a critical point.

Second Derivative Test for Local Extrema

$\boxed{\text{* critical point is either an extrema or an inflection point}}$

A *maximum* / *minimum* occurs at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$ (*concave down*) / $f''(c) > 0$ (*concave up*)

Note If $f''(c) = 0$ then the second-derivative test cannot be used, and the first-derivative test must be used.

p192 # 25-37, 43, 45, 47, 51

this is not $f(x)$

$f(x) = -2\cdot\frac{1}{4}x^4 + 9\cdot\frac{1}{3}x^3 + C$

$f(x) = -\frac{1}{2}x^4 + 3x^3 + C$

Example: Given that $f'(x) = -2x^3 + 9x^2$

a) Identify where any extrema occur

b) Determine intervals on which f is increasing and on which f is decreasing

c) Determine intervals where the curve is concave up and where the curve is concave down

d) Sketch a possible graph. Label any significant points.

a) extrema ; look for critical points

$f'(x) = 0$

$0 = -2x^3 + 9x^2$

$= x^2(-2x+9)$

$x = 0$ or 4.5

x	-1	0	1	4.5	5
y'	$+$	0	$+$	0	$-$

global max at $x = 4.5$

b) increasing on intervals $(-\infty, 0)$ $(0, 4.5)$

decreasing on interval $(4.5, \infty)$

c) $y'' = 0$?

$y' = -2x^3 + 9x^2$

$y'' = -6x^2 + 18x$

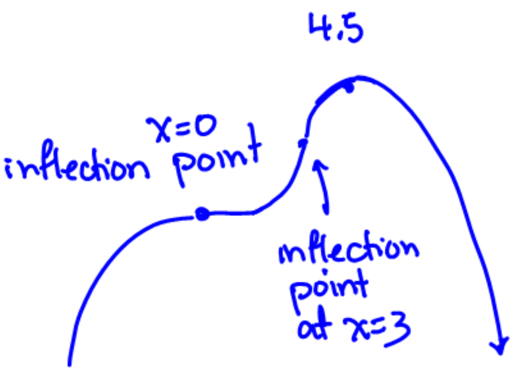
$0 = -6x^2 + 18x$

$= -6x(x-3)$

$x = 0$ $x = 3$

x	-1	0	1	3	4
y''	$-$	0	$+$	0	$-$
	conc down		conc up		conc down

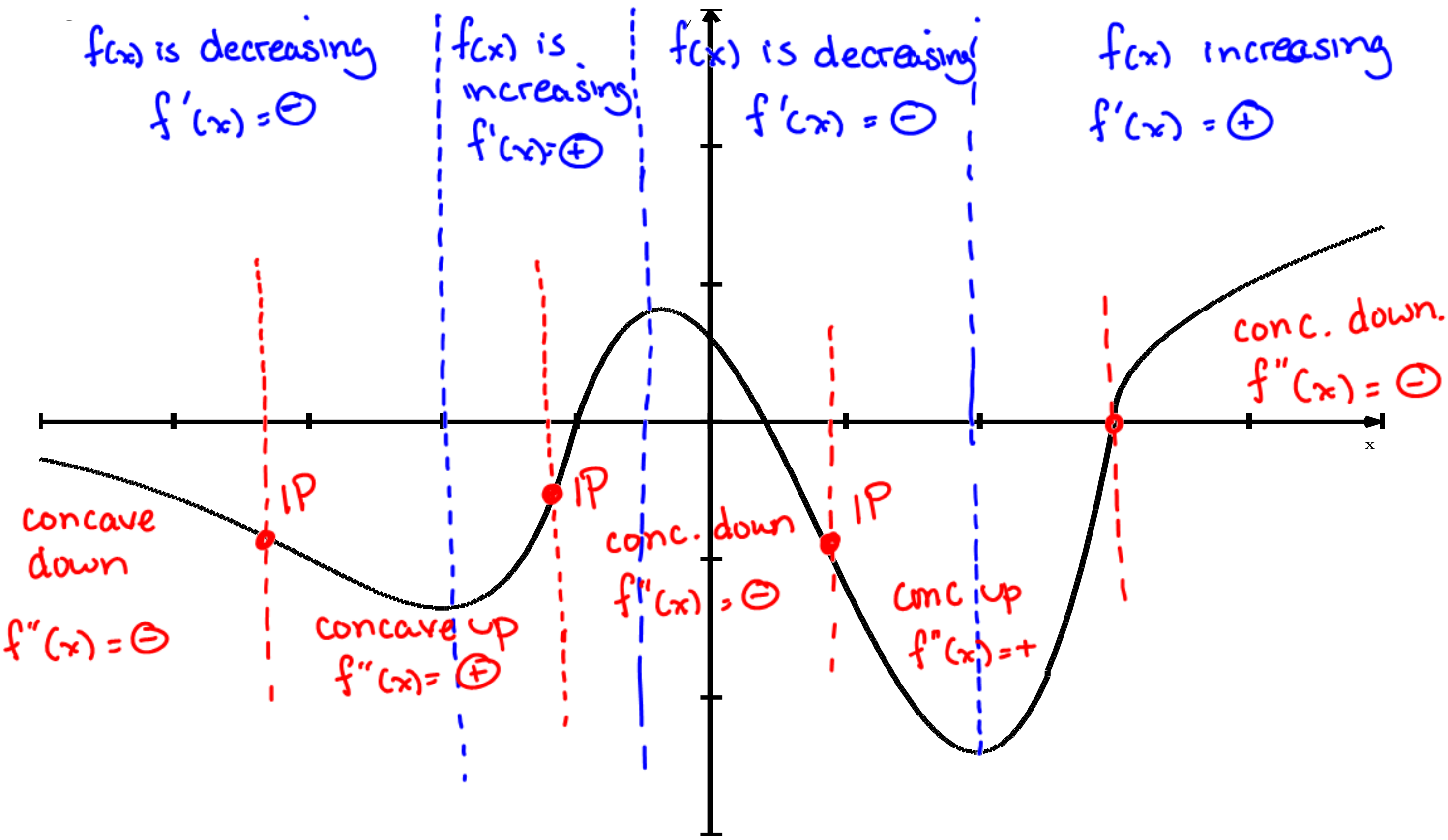
concavity is changing at $x = 0$, $x = 3$.
so these are inflection points



x		0		3		4.5	
y'	$+$	0	$+$	$+$		0	$-$
y''	$-$	0	$+$	0	$-$		$-$

Relationships between f, f' and f''

For the function graphed below indicate intervals where the function is increasing/decreasing, concave up/concave down, f' is positive/negative, f' is increasing/decreasing and f'' is positive/negative. Indicate the location of any critical points and inflection points.



$f'(x) = \oplus$ function is increasing

$f'(x) = \ominus$ function is decreasing

$f'(x) = 0$ critical point ; either extrema or inflection point

$f''(x) = \oplus$ concave up

$f''(x) = \ominus$ concave down

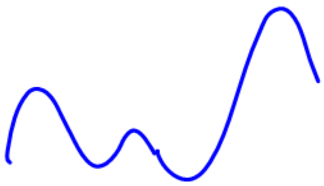
$f''(x) = 0$ inflection point.

Relationships Between Functions and their Derivatives

Differentiable

if f(x) is differentiable then it is continuous

eg



no corners, no cusps.

$y' > 0$

function is increasing

may be

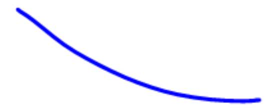
or

$y' < 0$

function is decreasing

may be

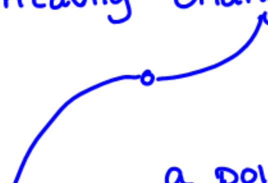
or



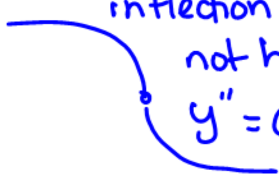
$y'' = 0$

point of inflection

concavity changes



or



a point of inflection may not have $y'' = 0$

$y'' > 0$

y' is increasing

y is concave up

$y'' < 0$

y' is decreasing

y is concave down.

y' changes sign

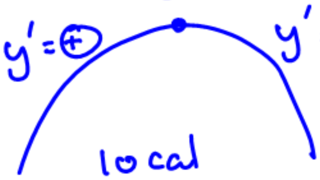
function changes direction

$\therefore$ extrema and a critical point

$y' = 0$ and $y'' < 0$

$y' = 0$

$y' = +$ $y' = -$

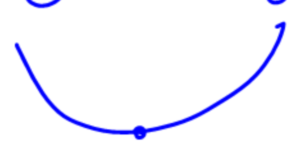


local maximum

conc down.

$y' = 0$ and $y'' > 0$

$y' = -$ $y' = +$



local minimum

concave up.

$\boxed{\text{p203 \#1-29}}$

Curve Sketching

Sketching Functions With Continuous Derivatives

3. Determine any x and /or y intercepts.

4. Determine any vertical or horizontal asymptotes.

5. Determine whether the function possesses any symmetries.

3. Determine the first and second derivatives of the function.

4. Determine all critical points by setting $f'(x) = 0$

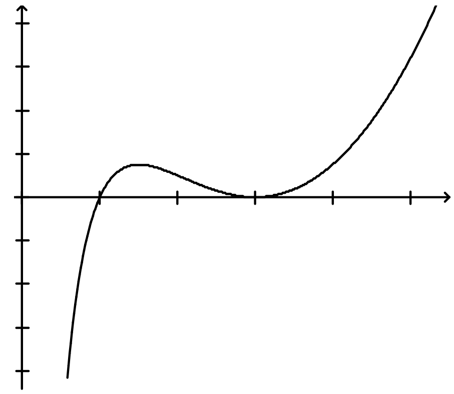
5. Use the second-derivative test to examine concavity at each of the critical points, and determine whether critical points are maximum or minimum. If $f''(x) = 0$, then the first-derivative test must be used.

6. Determine any possible inflection points by setting $f''(x) = 0$ and then test for sign change through the possible inflection point. If there is a sign change, then it is an inflection point; no sign change means that it is not an inflection point.

Plot intercepts, critical points and inflection points and then sketch the graph, taking care to show horizontal tangency and correct concavity at appropriate points.

Example: Sketch the graph of $y = (x-1)e^x$

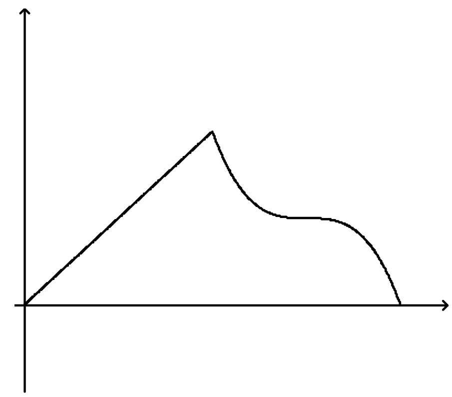
Curve Sketching Review



The graph of f' on $[0,5]$ is shown. Use this graph for #1 and 2.

1. f has a local minimum at x= ?

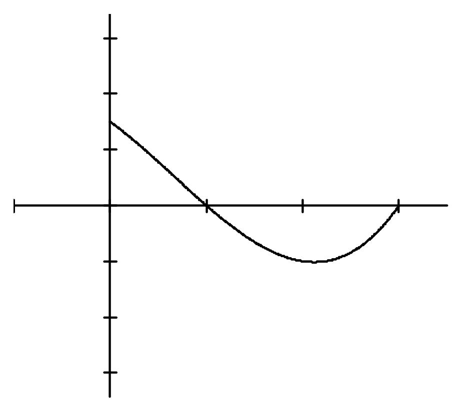
2. f has an inflection point at x= ?



3. It follows from the graph of f', shown at the right, that

   a) f is not continuous at $x = a$
   b) f is continuous but not differentiable at $x = a$
   c) f has a relative maximum at $x = a$
   d) f has a point of inflection at $x = a$
   e) none of these

4. A local minimum value of the function $y = \frac{e^x}{x}$ is ___________

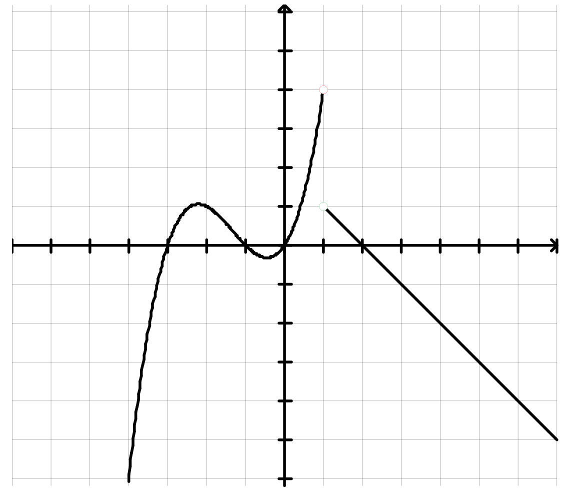


5. Given f' as graphed, sketch a possible graph for f

6. If $f(x) = xe^x$, then at $x = 0$
   a) f is increasing
   b) f is decreasing
   c) f has a relative maximum
   d) f has a relative minimum
   e) f' does not exist

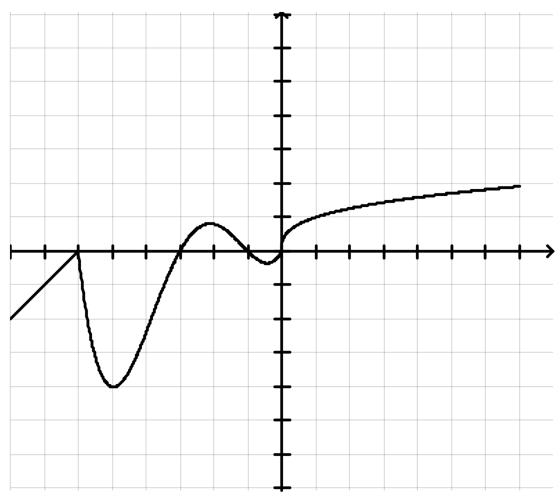
Unit 4.7 Warmup

1. Given the graph of $y = f'(x)$ to the right determine for the function $y = f(x)$
(Assume that the function $y = f(x)$ is a continuous function)



a) where any critical points occur, and whether they are maximum or minimum
b) intervals where $f(x)$ is increasing or decreasing
c) where any inflection points might occur
d) intervals where $f(x)$ is concave up; concave down

2. How many critical points and inflection point does the graph below appear to have?



3. Must all cubic function have *i*) a critical point? *ii*) an inflection point? Explain.

Unit 4.7 Warmup II

1. Sketch a graph of a function with the following properties:

$$f(x),\ f'(x),\ f''(x) \text{ are defined and continuous for all } x \neq 0$$

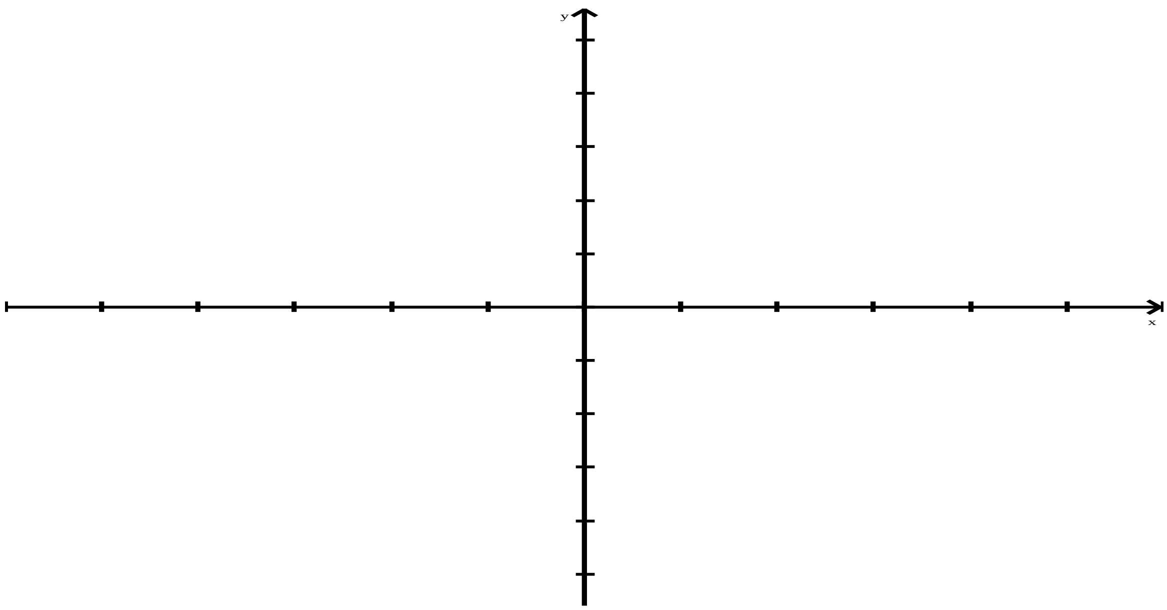
$$\lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = 2\ ,\ \lim_{x\to 0} f(x) = \infty$$

$$f(-3) = f(-1) = f(2) = 0 \qquad f'(-2) = f'(-4) = f'(1) = 0$$

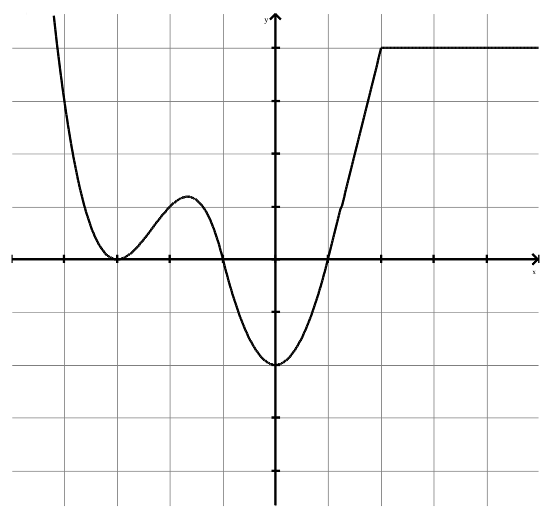
$$-3, -1, 2 \text{ are the only zeros of } f(x);\ \ -4, -2, 1 \text{ are the only zeros of } f'(x)$$

$$f''(x) < 0 \text{ for all } x \text{ in } (-5,-3)\cup(1,2)\ ;\ f''(-5) = f''(-3) = f''(1) = f''(2) = 0$$

$$f''(x) > 0 \text{ for all } x \text{ in } (-\infty,-5)\cup(-3,0)\cup(0,1)\cup(2,\infty)$$



2. Given the graph of $y = f'(x)$, determine intervals where $y = f(x)$ is increasing/decreasing and concave up/concave down. Also determine the *x* coordinates of any critical points or inflection points.



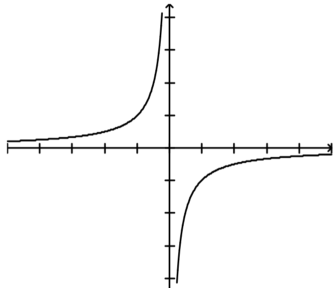
3. Find any extrema for the function $y = x^2 e^{\frac{1}{x}}$

Warmup 4.8

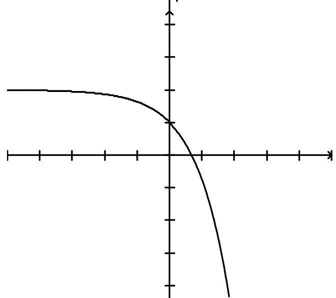
Match the functions with the graph which represents the derivative

Functions

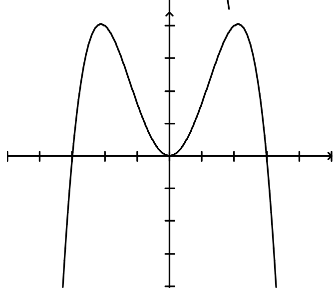
1)



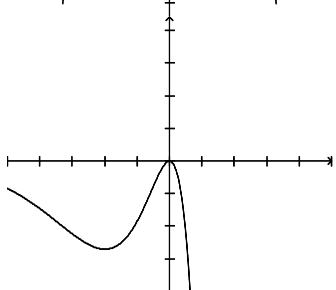
2)



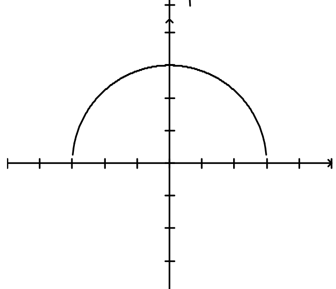
3)



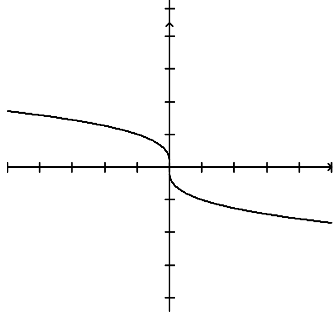
4)



5)

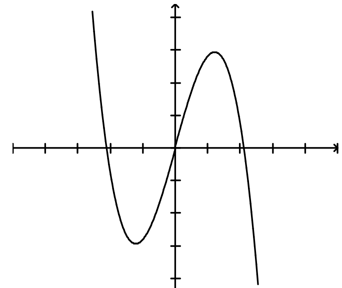


6)

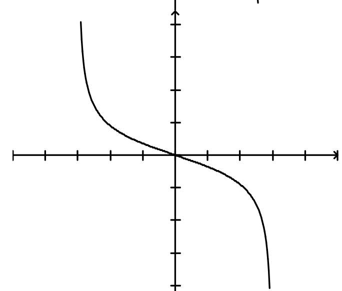


Derivatives

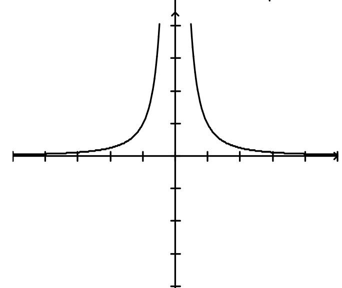
A)



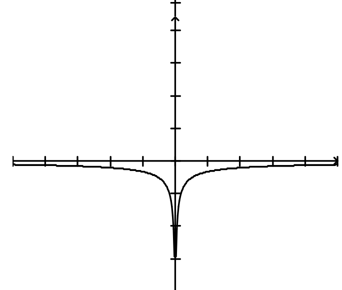
B)



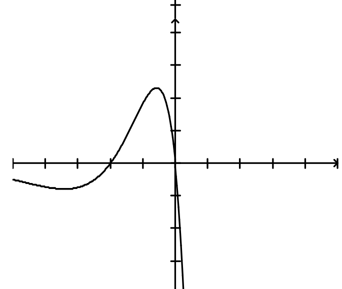
C)



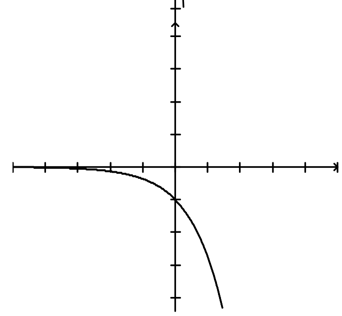
D)



E)

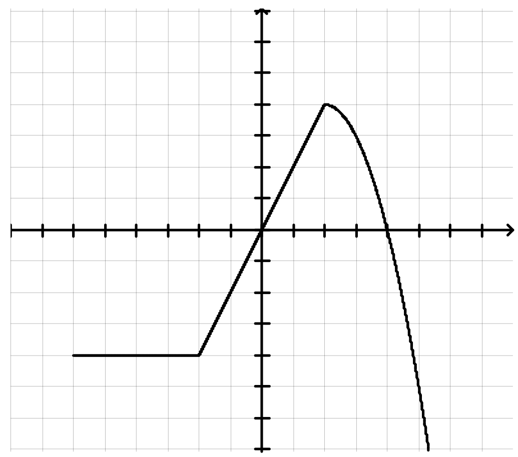


F)



2. A function has $y' = x^2 - 5x + 6$. Determine where any critical points and inflection points of the function exist. Also determine intervals of increasing, decreasing, concave up and concave down.

3. To the right is the graph of the $y = f'(x)$. Sketch a possible graph of $y = f(x)$



4. Determine the *x* values of any points on the graph $y = x^2 - 6x$ on $[1, 3]$ which satisfy the Mean Value Theorem.

5. A Calculus textbook is dropped from a 100 m cliff on the planet Concave, where the acceleration due to gravity is $2.4\ \text{m/s}^2$. What is the velocity of the book when it hits the ground?