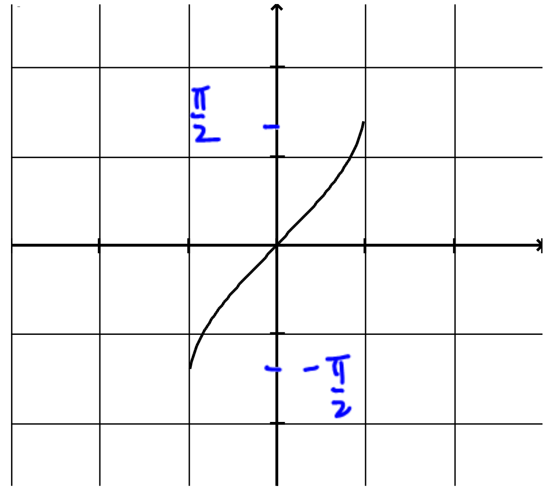
Derivatives of Inverse Trigonometric Functions

Recall the definition and the graph of $y = \sin^{-1} x$.



$\sin^{-1} x$ = the angle with a sine of x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Domain: $-1 \le x \le 1$ "value of sine"

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ "angle"

Also $y = \sin^{-1} x$ and $x = \sin y$ are the same equation provided $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

From the graph what appears to be true about the slopes of the tangents (derivative) of $y = \sin^{-1} x$?

it is always positive, it is undefined/vertical at $x = \pm 1$

What is $\frac{d}{dx}\sin^{-1} x$ or $\frac{d}{dx}\arcsin x$?

To differentiate, we will rewrite $y = \sin^{-1} x$ as $x = \sin y$ and use implicit differentiation to find $\frac{dy}{dx}$.

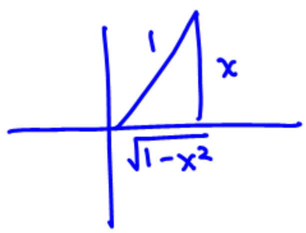
$x = \sin y$

$\frac{d(x)}{dx} = \frac{d(\sin y)}{dx}$

$1 = \cos y \cdot y'$

$y' = \frac{1}{\cos y}$

$y' = \frac{1}{\cos(\sin^{-1}(x))}$

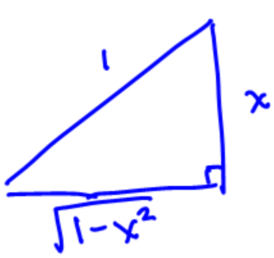


$y' = \frac{d(\sin^{-1}(x))}{dx} = \frac{1}{\sqrt{1-x^2}}$

or if x is a function, apply chain rule

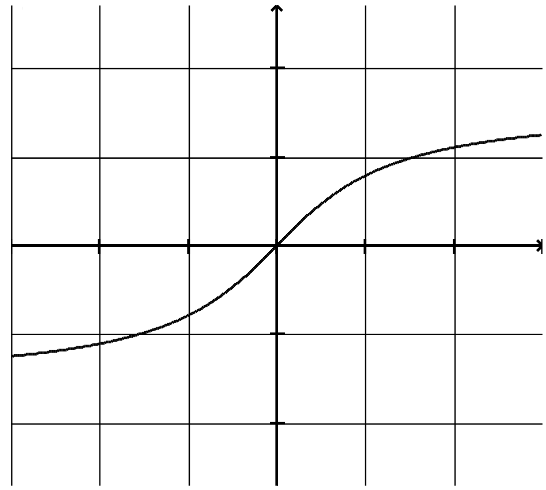
$\frac{d(\sin^{-1}(u))}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

To find the derivative of $\cos^{-1} x$, we can use the identity: $\cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$



$\frac{d(\cos^{-1} x)}{dx} = \frac{d\left(\frac{\pi}{2} - \sin^{-1}(x)\right)}{dx} = \frac{-1}{\sqrt{1-x^2}}$

Similarly, we can differentiate $y = \tan^{-1} x$



Domain: $\{\text{Reals}\}$

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

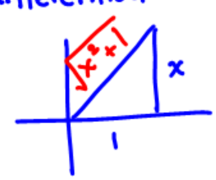
What appears to be true about the slopes of the tangents (derivative) for this function?

always positive. max value of derivative at $x=0$

$y = \tan^{-1} x$ is the same as $x = \tan y$ provided $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \tan^{-1} x$

$x = \tan y$ ← implicit differentiation



$1 = \sec^2(y) \cdot y'$

$y' = \frac{1}{\sec^2 y}$

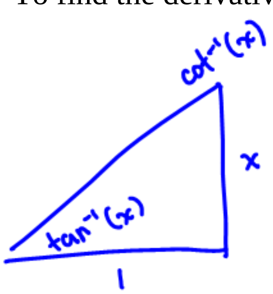
$y' = \cos^2 y$

$y' = \cos^2(\tan^{-1} x)$

$y' = \left(\frac{1}{\sqrt{x^2+1}}\right)^2$

$y' = \frac{1}{x^2+1}$

To find the derivative of $\cot^{-1} x$, we can use the identity: $\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$

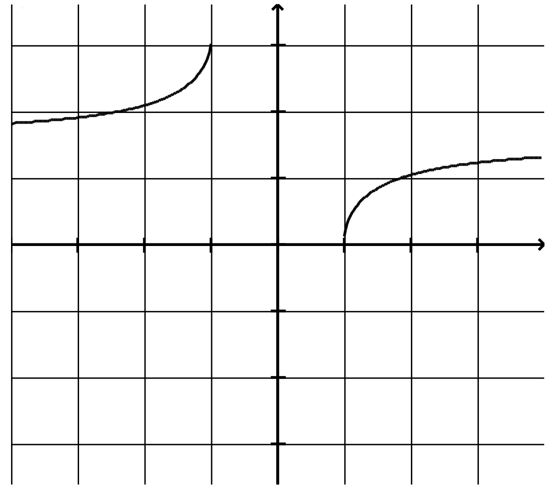


$$\frac{d(\cot^{-1}(x))}{dx} = \frac{d\left(\frac{\pi}{2} - \tan^{-1}(x)\right)}{dx}$$

$$= 0 - \frac{1}{x^2+1}$$

$$\frac{d(\cot^{-1}(x))}{dx} = \frac{-1}{x^2+1}$$

Lastly, we can differentiate $y=\sec^{-1} x$



Domain: $x \geq 1$ or $x \leq -1$

Range: $0 \leq y \leq \pi$ $y \neq \frac{\pi}{2}$

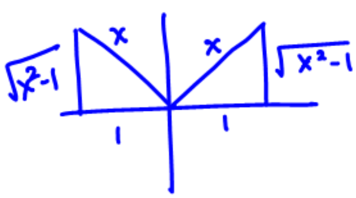
What appears to be true about the slopes of the tangents (derivative) for this function?

slope is always + but undefined at $x = \pm 1$

$y=\sec^{-1} x$ is the same as $y=\sec x$ provided $0 \leq y \leq \pi$

$x = \sec y$ implicit differentiation

$1 = \sec y \cdot \tan y \cdot y'$



$$y' = \frac{1}{\sec y \tan y}$$

$$y' = \frac{1}{\sec(\sec^{-1} x) \cdot \tan(\sec^{-1} x)}$$

$$y' = \frac{1}{|x| \cdot \sqrt{x^2-1}}$$

tan could be ± depending on quadrant.

$$\frac{d(\sec^{-1}(x))}{dx} = \frac{1}{|x| \cdot \sqrt{x^2-1}}$$

To find the derivative of ~~$\cot^{-1}$~~ x, we can use the identity: $\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$

$\csc^{-1} x$

* same reasoning as $\sin^{-1}(x)$ and $\cos^{-1}(x)$

and

$\tan^{-1}(x)$ and $\cot^{-1}(x)$

$$\frac{d(\csc^{-1} x)}{dx} = \frac{-1}{|x| \cdot \sqrt{x^2-1}}$$

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p 162 #1-15

#2 $y = \cos^{-1}\left(\frac{1}{x}\right)$ $\frac{d(\cos^{-1} u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

$y' = \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot -1(x)^{-2}$

$y' = \frac{1}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}} = \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}} = \frac{1}{x^2 \cdot \frac{\sqrt{x^2-1}}{x}} = \frac{1}{x\sqrt{x^2-1}}$

#4 $y = \sin^{-1}(1-t) = \frac{1}{\sqrt{1-(1-t)^2}} \cdot (-1) = \frac{-1}{\sqrt{1-(1-t)^2}}$