

## Warmup 2.8

Determine the following limits:

$$1. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$2. \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \quad (\text{Hint: try multiplying by the conjugate})$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} &= \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)} \\ &= \frac{-1(1 - \cos^2 x)}{x(\cos x + 1)} = \frac{-1(\sin^2 x)}{x(\cos x + 1)} \end{aligned}$$

$$\begin{aligned} 3. \lim_{h \rightarrow 0} \frac{\frac{1}{(x+5+h)^4} - \frac{1}{(x+5)^4}}{h} &= -1 \cdot \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1} \\ &= -1(1) \cdot \frac{0}{2} \\ &= 0 \end{aligned}$$

$$4. \lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3}$$

5. Use your calculator to graph the derivative of  $y = \sin x$ . What does the graph suggest that the derivative of  $\sin x$  might be?

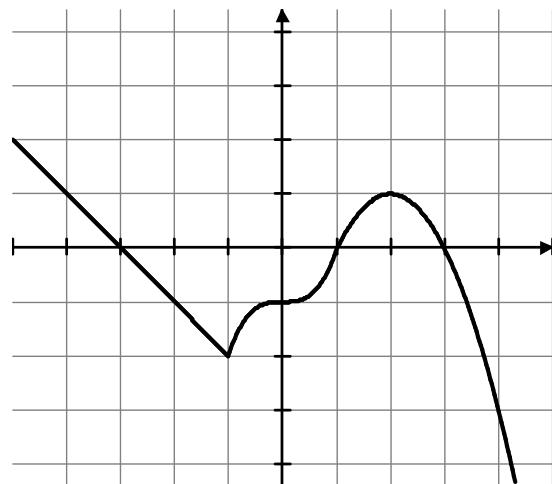
6. Determine the following derivatives:

a) $\frac{d}{dx} 18x - 12$	b) $\frac{d}{dx} 5x^3 - 12x + 19$	c) $\frac{d}{dx} \frac{2x-14}{5x-8}$
d) $\frac{d}{dx} \frac{12}{x^7}$	e) $\frac{d}{dx} \sqrt{x} - \frac{3}{x}$	f) $\frac{d^2}{dx^2} 4x^3 - 5x^2 + 12x - 6$

7. Given the following graph:

- a) If this graph represented the original function, at how many points would the derivative be undefined?

At how many points would the derivative equal zero?



- b) If this graph represented the derivative, at how many points would the original function have the derivative undefined?

At how many points would the original function have a horizontal tangent?

## Derivatives of Trigonometric Functions

To find  $\frac{d \sin x}{dx}$  and  $\frac{d \cos x}{dx}$  we use the definition of the derivative and the previously established

limits:  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  and  $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$\frac{d \sin x}{dx}$	$= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$	
	$= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$	
	$= \lim_{h \rightarrow 0} \frac{(\sin x \cos h - \sin x) + \sin h \cos x}{h}$	
	$= \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1) + \sin h \cos x}{h}$	
	$= \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \rightarrow 0} \frac{\sin h \cos x}{h}$	
	$= \sin x(0) + 1(\cos x)$	
$\frac{d(\sin x)}{dx}$	$= \cos x$	

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$\frac{d \cos x}{dx}$	$= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}$	$= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$
	$= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$	
	$= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h}$	
	$= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1)}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h}$	
	$= \cos x(0) - \sin x(1)$	

$$\frac{d \cos x}{dx} = -\sin x$$

Using identities and differentiation rules, the derivatives of  $\tan x$  and the reciprocal trigonometric functions can then be determined.

$$\frac{d}{dx} \tan x = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) = \frac{(\cos x)(\cos x) - (-\sin x)(\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \boxed{\frac{1}{\cos^2 x} \text{ or } \sec^2 x}$$

$$\frac{d}{dx} \csc x = \frac{d}{dx} \left( \frac{1}{\sin x} \right) = \frac{(0)(\sin x) - (\cos x)(1)}{\sin^2 x}$$

$$\boxed{\frac{d(\csc x)}{dx} = \frac{-\cos x}{\sin^2 x}} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x} = \boxed{-\cot x \cdot \csc x}$$

$$\frac{d}{dx} \sec x = \frac{d}{dx} \left( \frac{1}{\cos x} \right) = \frac{(0)(\cos x) - (-\sin x)(1)}{\cos^2 x}$$

$$\frac{d}{dx} (\sec x) = \frac{\sin x}{\cos^2 x} \quad \text{or} \quad \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \cdot \sec x$$

$$\frac{d}{dx} \cot x = \frac{d}{dx} \left( \frac{\cos x}{\sin x} \right) = \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= -1 \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} \text{ or } -\csc^2 x$$

Thus the following have been determined:

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d(8 \cdot f(x))}{dx} = 8 \cdot \frac{df(x)}{dx}$$

Determine  $\frac{dy}{dx}$  for each of the following:

$$1. \quad y = 8 \sin x \quad = (0)(\sin x) + (\cos x)(8)$$

$$y' = 8 \cos x$$

$$2. \quad y = x \cos x$$

use product

$$y' = (1)(\cos x) + (-\sin x)(x)$$

$$y' = \cos x - x \cdot \sin x$$

$$3. \quad y = \sin x \cos x$$

use product

$$y' = (\cos x)(\cos x) + (-\sin x)(\sin x)$$

$$y' = \cos^2 x - \sin^2 x$$

$$4. \quad y = \frac{1}{x} - \tan x$$

$$y = x^{-1} - \tan x$$

$$y' = -1 \cdot x^{-2} - \sec^2 x$$

$$y' = \frac{-1}{x^2} - \sec^2 x$$

$$5. \quad y = \frac{\cos x}{1 + \sin x}$$

quotient rule

$$= \frac{(-\sin x)(1 + \sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2}$$

$$6. \quad \text{Determine the equation of the tangent to } y = 4 \cos x \text{ at } x = \frac{\pi}{2}$$

$$y'(\frac{\pi}{2}) = 4(-\sin x) \Big|_{x=\frac{\pi}{2}} \quad y(\frac{\pi}{2}) = 4 \cos(\frac{\pi}{2})$$

$$= 4(-1) \quad y(\frac{\pi}{2}) = 0$$

$$= -4 \quad = \frac{-\sin x - 1}{(1 + \sin x)^2}$$

$$y - 0 = -4(x - \frac{\pi}{2}) \quad = \frac{-1(\sin x + 1)}{(1 + \sin x)^2}$$

$$y = -4(x - \frac{\pi}{2}) \quad = \frac{-1}{1 + \sin x}$$

7. Determine the  $x$  coordinates of any points on the graph of  $y = \sin x$  where there is a horizontal tangent. when  $y' = 0$

$$y' = \cos x$$

$$\cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$x = \frac{\pi}{2} + n\pi \quad n \text{ is an integer.}$$

8. Determine:  $\frac{d^9}{dx^9} \cos x$

$$y' = -\sin x$$

$$y'' = -\cos x$$

$$y''' = \sin x$$

$$y^4 = \cos x$$

$$y^5 = -\sin x$$

$$y^6 = -\cos x$$

$$y^7 = \sin x$$

$$y^8 = \cos x$$

$$\frac{d^9}{dx^9} (\cos x) = -\sin x$$

9. Determine the exact value of  $\lim_{h \rightarrow 0} \frac{\sin(\frac{\pi}{3} + h) - \sin(\frac{\pi}{3})}{h} = \frac{d(\sin x)}{dx} \Big|_{x=\frac{\pi}{3}}$



$$= \cos x \Big|_{x=\frac{\pi}{3}}$$

$$= \cos\left(\frac{\pi}{3}\right) = \boxed{\frac{1}{2}}$$

10. What is the maximum slope on the graph  $y = 8\sin x$  and where does it occur?

$$y' = 8\cos x \quad \text{where does } \cos x \text{ have a max value?}$$

$$x = 0, 2\pi, 4\pi$$

$$\text{max value} = 8(\cos 0)$$

$$= 8$$

$$\text{occurs at } x = 0 + n2\pi$$

## Derivatives

**Definition of the Derivative:**  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$        $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$

**Derivative of a Constant**

$$\frac{d}{dx} k = 0$$

**Power Rule**

$$\frac{d}{dx} x^n = n x^{n-1}$$

**Constant times a Function**

$$\frac{d}{dx} k \cdot f(x) = k \cdot f'(x)$$

**Product Rule**

$$\frac{d}{dx} f \cdot g = f' g + f g'$$

**Quotient Rule**

$$\frac{d}{dx} \frac{f}{g} = \frac{f' g - g' f}{g^2}$$

**Chain Rule:**  $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$       or       $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

**Derivatives of Trigonometric Functions:**

$$\frac{d}{dx} \sin u = \cos u \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \csc u = -\csc u \cot u \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \cos u = -\sin u \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \sec u = \sec u \tan u \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \tan u = \sec^2 u \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \cot u = -\csc^2 u \cdot \frac{du}{dx}$$

**Derivatives of Inverse Trigonometric Functions**

$$\frac{d}{dx} \sin^{-1} u = \frac{d}{dx} \arcsin u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \quad |u| < 1$$

$$\frac{d}{dx} \cos^{-1} u = \frac{d}{dx} \arccos u = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \quad |u| < 1$$

$$\frac{d}{dx} \tan^{-1} u = \frac{d}{dx} \arctan u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \cot^{-1} u = \frac{d}{dx} \operatorname{arc cot} u = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \sec^{-1} u = \frac{d}{dx} \operatorname{arc sec} u = \frac{1}{|u| \cdot \sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \csc^{-1} u = \frac{d}{dx} \operatorname{arc csc} u = \frac{-1}{|u| \cdot \sqrt{u^2 - 1}} \cdot \frac{du}{dx}$$

**Exponential Functions:**  $\frac{d}{dx} e^u = e^u \frac{du}{dx}$

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

**Logarithmic Functions:**  $\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$

$$\frac{d}{dx} \log_b u = \frac{1}{u \ln b} \frac{du}{dx}$$