Examples

1. Determine the derivative of $y = \frac{1}{x^5}$ by

a) Using the quotient rule	b) rewriting as a negative power and then using the power rule

2. Determine the equation of the tangent to $y = \frac{5x+1}{3x-1}$ at $x = 1$

3. What is $\lim_{h \to 0} \frac{\frac{8}{(x+h)^{12}} - \frac{8}{x^{12}}}{h}$

Higher Order Derivatives

The process of finding the derivative of a function can also be applied to the derivative itself. (ie we can find the derivative of the derivative). y' or $\frac{dy}{dx}$ is called the first derivative of *y* with respect to *x*.

The second derivative with respect to *x* is $y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

The third derivative is $y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$ and the n^{th} derivative of *y* with respect to *x* is $y^{(n)} = \frac{d^ny}{dx^n}$

Example: If $y = x^3 + 5x^2 - 7x - 18$, what is $\frac{d^2y}{dx^2}$?

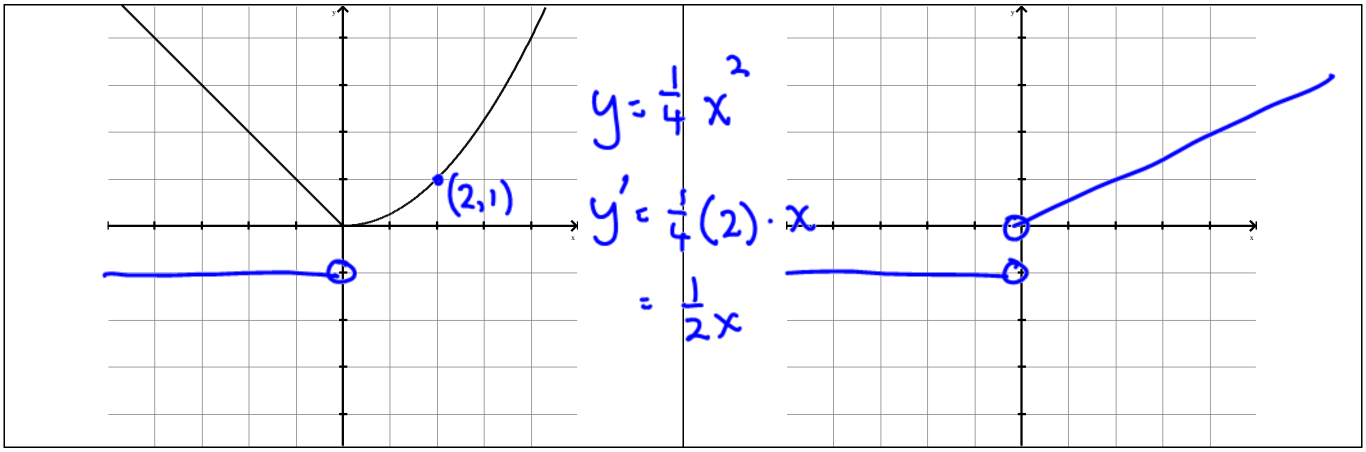
$\frac{dy}{dx} = y' = 3x^2 + 10x - 7$

$\frac{d^2y}{dx^2} = y'' = 6x + 10$

Warmup 2.5

$y = \left(\frac{1}{2}x\right)^2 = \left(\frac{1}{2}x\right)\left(\frac{1}{2}x\right)$

1. Sketch a graph of the derivative of the following function



$y = \frac{1}{4}x^2$

$y' = \frac{1}{4}(2) \cdot x$

$= \frac{1}{2}x$

2. Find the derivatives of the following functions

product rule

a) $y = (3x - 4)(x^2 - 7x + 3)$

$y' = f'g + g'f$

$= (3)(x^2 - 7x + 3) + (2x - 7)(3x - 4)$

$= 3x^2 - 21x + 9 + 6x^2 - 8x - 21x + 28$

$= 9x^2 - 50x + 37$

b) $f(x) = \dfrac{4x^3}{x^2 + 3}$

quotient rule

$y' = \dfrac{f'g - g'f}{g^2}$

$= \dfrac{(12x^2)(x^2+3) - (2x)(4x^3)}{(x^2+3)^2}$

$= \dfrac{12x^4 + 36x^2 - 8x^4}{(x^2+3)^2} = \dfrac{4x^4 + 36x^2}{(x^2+3)^2}$

c) $y = \frac{1}{f(x)}$

use quotient rule

$$= \frac{f'g - g'f}{g^2} = \frac{(0)\cdot f(x) - f'(x)\cdot(1)}{f(x)^2} = \frac{-f'(x)}{f(x)^2}$$

$y = f(x)^{-1}$

you cannot do this without "chain rule"

d) $y = \sqrt{x}$

$y = x^{\frac{1}{2}}$

$$y' = \frac{1}{2}x^{-\frac{1}{2}} \rightarrow \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$$

$$y' = \frac{1}{2\sqrt{x}}$$

3. Suppose g and h are differentiable functions at $x = 0$ and that $g(0) = 5,\ g'(0) = -2$ $h(0) = -3,\ h'(0) = 4$. Find the value of the following derivatives at $x = 0$

a) $\frac{d}{dx}(gh)$

$$= g'h + h'g \Big|_{x=0}$$

$$= (-2)(-3) + (4)(5)$$

$$= 26$$

b) $\frac{d}{dx}(5g - 9h)$

$$= 5g' - 9h' \Big|_{x=0}$$

$$= 5(-2) - 9(4)$$

$$= -46$$

c) $\frac{d}{dx}\left(\frac{g}{h}\right)$

$$= \frac{g'h - h'g}{h^2}\Bigg|_{x=0}$$

$$= \frac{(-2)(-3) - (4)(5)}{(-3)^2} = \frac{-14}{9}$$

Rules for Differentiation - Tangent Lines

6. Find the equation of the tangent line to the curve $y = 2x^2 - 6x + 7$ when $x = 2$.

$y' \big|_{x=2}$

$y(2) = 2(2)^2 - 6(2) + 7$

$= 8 - 12 + 7$

$= 3$

$y' = 4x - 6 \big|_{x=2}$

$y'(2) = 4(2) - 6$

$= 2$

$$\boxed{y - 3 = 2(x-2)}$$

7. Find all points on the curve $y = x^3 + 2x^2 + x - 7$ where the tangent line is horizontal.

where is $y' = 0$

$y' = 3x^2 + 4x + 1$

$0 = 3x^2 + 4x + 1$

$0 = (3x+1)(x+1)$

$x = \frac{-1}{3}$ or -1

$f(-\frac{1}{3}) = \frac{-1}{27} + 2\left(\frac{1}{9}\right) + \left(\frac{-1}{3}\right) - 7$

$\left(-\frac{1}{3}, \frac{-193}{27}\right)$

$f(-1) = -7$

$(-1, -7)$

8. Find all points on the curve $y = \frac{1}{x}$ where the slope of the normal is 4.

$m_{tan} = \frac{-1}{4}$

$y = x^{-1}$

$y' = -x^{-2}$

$y' = -\frac{1}{x^2}$

$\frac{-1}{4} = \frac{-1}{x^2}$

$4 = x^2$

$x = \pm 2$

$x = 2 \quad f(2) = \frac{1}{2}$

$$\boxed{\left(2, \tfrac{1}{2}\right)}$$

$x = -2 \quad f(-2) = \frac{-1}{2}$

$$\boxed{\left(-2, -\tfrac{1}{2}\right)}$$

9. Find the tangent to the curve $y = \frac{8}{4 + x^2}$ at the point (2 , 1)

f, g

quotient rule.

$$y' = \frac{f'g - g'f}{g^2}$$

$$= \left. \frac{(0)(4+x^2) - (2x)(8)}{(4+x^2)^2} \right|_{x=2}$$

$$\boxed{y-1 = -\frac{1}{2}(x-2)}$$

$$y'(2) = \frac{-2(2)(8)}{[4+(2)^2]^2} = \frac{-32}{64} = -\frac{1}{2}$$

10. If gas in a cylinder is to be maintained at a constant temperature T, the pressure P is related to the volume V by the formula:

$$P = \frac{nRT}{V - nB} - \frac{an^2}{V^2}$$

where a, b, n, and R are constants. Find $\frac{dP}{dV}$ and describe what it means.

$\frac{dP}{dV}$: how fast is P changing as V changes.

$$P = \frac{nRT}{V-nB} - \frac{an^2}{V^2}$$

$$\frac{an^2}{V^2} = an^2 V^{-2}$$

$$\frac{dP}{dV} = \frac{f'g - g'f}{g^2} -$$

$$= \frac{0(V-nB) - (1)(nRT)}{(V-nB)^2} - -2an^2V^{-3}$$

$$\frac{dP}{dV} = \frac{-nRT}{(V-nB)^2} + \frac{2an^2}{V^3}$$

2.6 Warmup

1. Determine $\dfrac{d}{dx}\left(\dfrac{fg}{h}\right)$

2. Determine $\dfrac{d}{dx} f^3$ (Hint: Treat f^3 as $f \bullet f \bullet f$ and use the product rule)

3. Determine $\dfrac{d}{dt}\left(\dfrac{At-1}{A^2 t}\right)$ where A is a constant.

4. If $y = x^n$ what is $y^{(n)}$ or $\dfrac{d^n y}{dx^n}$ (or what is the n^{th} derivative)?

5. y' If you keep on differentiating a polynomial function, will the derivative eventually become zero? Why or why not?

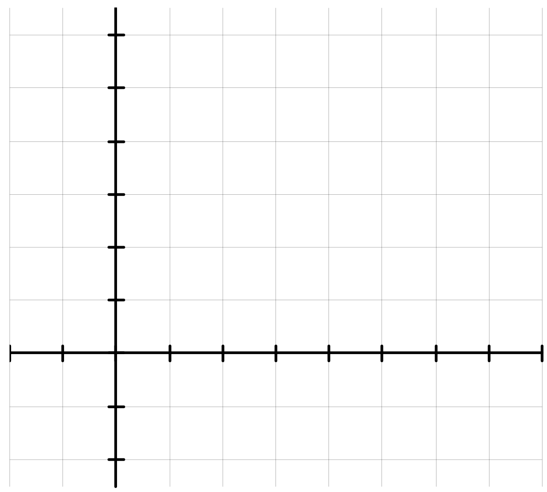
Velocity and Acceleration

For any function $s = f(t)$ where s = distance (or position or displacement) and t = time, $\frac{\Delta s}{\Delta t}$ is the average velocity, s' is the instantaneous velocity $v(t)$, and s'' is the instantaneous acceleration. Note that the average aceleration is $\frac{\Delta v}{\Delta t}$ and that $s''(t) = \frac{d^2 s}{dt^2} = v'(t) = \frac{dv}{dt} = a(t)$. Also speed is defined to be $|v(t)|$

Part One: **Straight Line Motion**

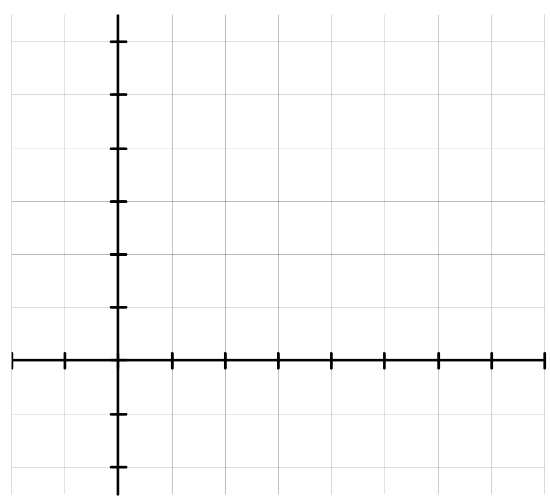
Example: A particle moves along a horizontal line according to $s = t^3 - t^2 - t + 1$ where s is the distance in centimeters and t is the time in seconds. Assume $t \geq 0$.

a) Sketch the graph of $s = f(t)$



b) Find the average velocity during the first 3 seconds.

c) Find the instantaneous velocity at time t and sketch the graph of the velocity function.



d) Find the velocity at $t = 2$ s

e) When is the velocity zero?

f) When is the particle moving to the left?

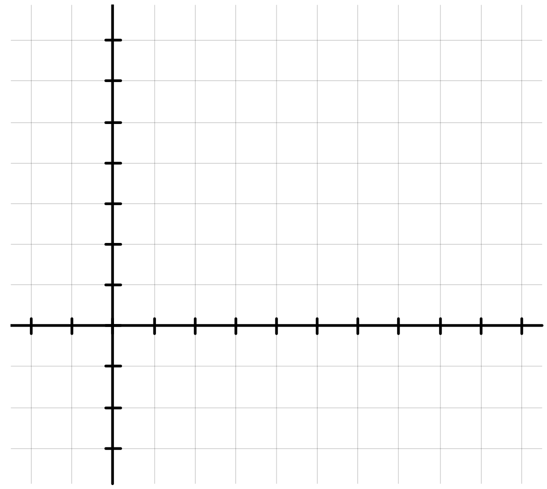
g) What is the acceleration at time t? At $t = 2$ s?

h) What is the acceleration when the velocity is a minimum?

Part Two: **Projectile Motion**

Example: A projectile is fired into the air from the top of a 20 m high hill. Its initial velocity is 50 m/s and its height after t seconds is given by the function $h = -5t^2 + 50t + 20$.

a) Sketch the graph of $h = f(t)$



b) Find the velocity at time t. At $t = 2$ s.

c) How long will it take to reach its maximum height?

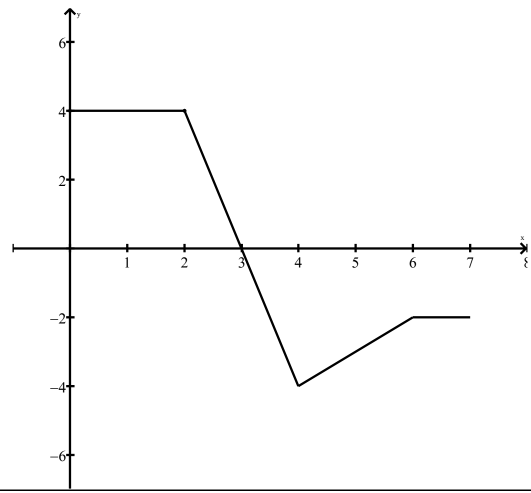
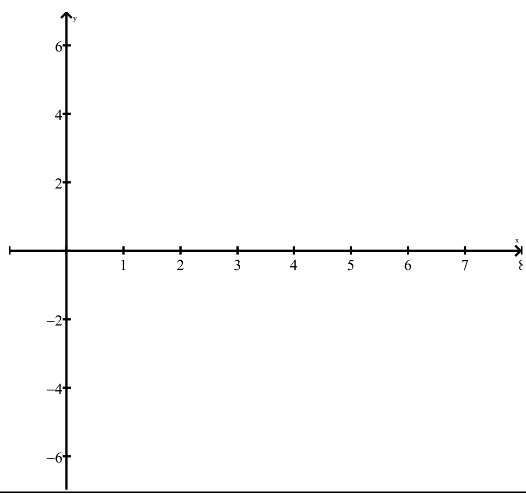
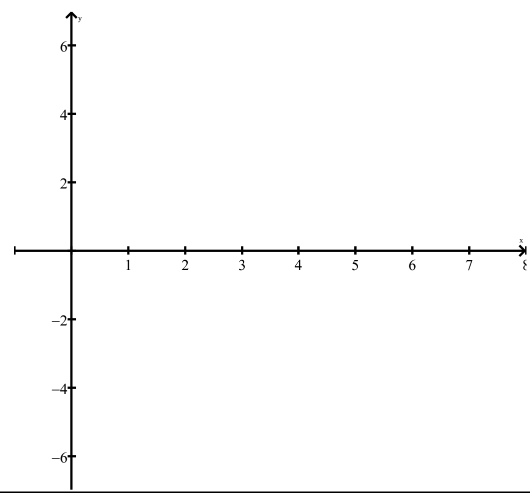
d) What is the maximum height reached?

e) What is the acceleration at time t ?

f) What is the projectile's velocity when it hits the ground?

Velocity, Speed and Acceleration

1. The graph shows the position $s(t)$ of a particle moving along a horizontal coordinate axis.

a) When is the particle moving to the left? b) When is the particle moving to the right? c) When is the particle standing still? d) Graph the particle's velocity and speed. e) When is the particle moving fastest?	Position (in.) 
Velocity (in./sec) 	Speed (in./sec) 

2. A particle moves along a vertical coordinate axis so that its position at any time $t \geq 0$ is given by the function $s(t) = \frac{1}{3}t^3 - 3t^2 + 8t - 4$ where *s* is measured in centimetres and *t* is measured in seconds.

a) Find the displacement during the first 6 seconds.

b) Find the average velocity during the first 6 seconds.

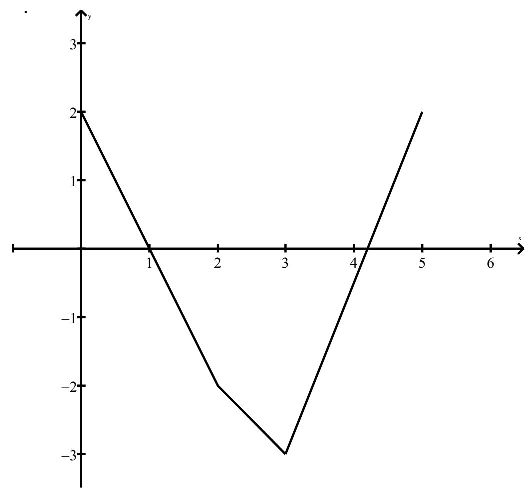
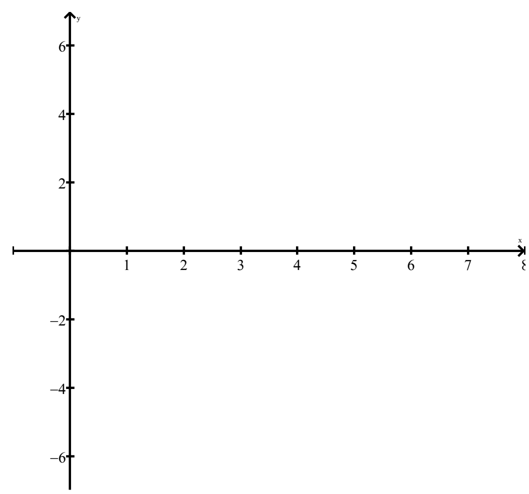
c) Find expressions for the velocity and acceleration.

d) For what values of *t* is the particle moving downward?

3. The graph shows the velocity $v = f(t)$ of a particle moving along a horizontal coordinate axis.

<table>
<tr>
<td>a) When does the particle reverse direction?

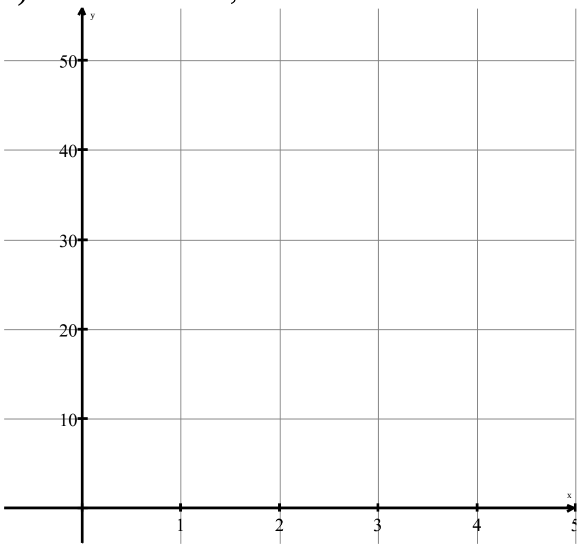
b) When is the particle moving at a constant speed?

c) When is the particle moving at its greatest speed?</td>
<td>Velocity (m/sec.)
</td>
</tr>
<tr>
<td>d) Graph the particle's acceleration. (where defined)</td>
<td>Acceleration (m/sec^2)
</td>
</tr>
</table>

4. The values of the coordinates *s* of a moving body for various values of *t* are given below.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (m)	40.0	35.0	30.2	36.0	48.2	45.0	38.2	16.0	0.2

a) Plot *s* versus *t*, and sketch a smooth curve through the given points.



b) Estimate the velocity at $t = 0.5$ sec and at $t = 2.5$ sec.

c) At what approximate value of *t* does the particle change direction?

d) At what approximate value of *t* is the particle moving at the greatest speed?

Applications

11. A bacteria culture has a population, n , after t hours given by $n = 400 + 280t + 49t^2$

a) Find $\dfrac{dn}{dt}$ and interpret the result.

b) Find the rate of growth after 3 hours.

12. In manufacturing, the cost of production $c(x)$ is a function of x, the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to the level of production and can thus be denoted as $\dfrac{dc}{dx}$. This is sometimes loosely defined to be the extra cost of producing 1 more unit, and is acceptable if the slope of c is not quickly changing near x.

The cost in dollars of producing x DVD drives is $c(x) = (50 - .1x)(40 + x)$.

a) What is the average cost of producing 100 DVD drives ?

b) What is the marginal cost of producing the 100th DVD drive ?

c) What is the cost of producing the 100th DVD drive ?

13. A car burns gas at a rate of $g(x) = \left(\dfrac{1000 + x^2}{200x}\right)$ liters per kilometer when traveling at x km/h

a) Graph $y = g(x)$

b) What is $g'(x)$ and how do you interpret it ?

c) Determine $i)$ $g'(60)$ $ii)$ $g'(80)$ $iii)$ $g'(110)$

d) What is the most efficient speed for fuel consumption ?