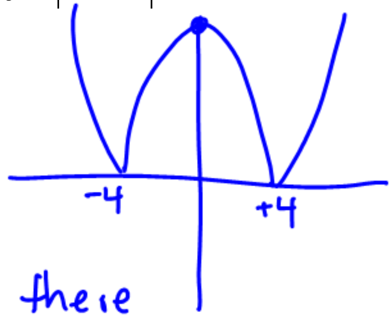
Warmup 2.4

$|-x^2+16)$

1. Use your calculator to determine the derivative of $y=\left|16-x^2\right|$ at $x=4$. Is this a reasonable value for the derivative at this point? Why or why not?

nDeriv(abs(16-x²), x, 4)



this is not differentiable at x = ±4 because there is a corner.

2. Graph the functions $f(x)=x^2$, $g(x)=\text{nDeriv}\left(x^2, x, x\right)$ and $h(x)=\left(f(x+.001)-f(x-.001)\right)/.002$.

What function does the graph of $g(x)$ appear to represent? Does this make sense?

g(x) represents a linear function

How does the graph of $h(x)$ compare to the graph of $g(x)$?

$$y_3=\frac{(x+.001)^2-(x-.001)^2}{.002}$$

it is very close, I can't tell the two apart.

3. Predict the answers to the following:

a) $\dfrac{d}{dx}5$ b) $\dfrac{d}{dx}x$ c) $\dfrac{d}{dx}x^2$

d) $\dfrac{d}{dx}x^3$ e) $\dfrac{d}{dx}x^4$ f) $\dfrac{d}{dx}x^{27}$

4. Use limits to verify your answer to "3e".

Rules for Differentiation

Derivative of a Constant Function

If $f(x) = k$ then $f'(x) = 0$

eg $f(x) = 3$

Power Rule for Positive Integer Powers

If $f(x) = x^n$ then $f'(x) = n \cdot x^{n-1}$

Constant Multiple of a Function

If $f(x) = k \bullet g(x)$ then $f'(x) = k \cdot g'(x)$

$f(x) = 5 \cdot x^3$ $\quad f'(x) = 5 \cdot (3x^2)$

Sum and Difference Rule

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

Product Rule

$$\frac{d}{dx}(f(x) \bullet g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \_\_\_\_\_\_\_\_\_\_$$

The Power Rule for Derivatives

If $y = x^n$ then $y' = n\,x^{n-1}$ for all real values of n

This can be proved for all positive integers n in the following way

Proof : The factor theorem says: If a is a root of $f(x) = 0$ then $x - a$ is a factor of $f(x)$

If $f(x) = x^n - a^n$ then $f(a) =$ 0 and consequently $(x-a)$ is a factor of $f(x)$

if $f(x) = x^n - a^n$

$f(a) = a^n - a^n$

By observing the pattern

$$x - a = x - a$$

$$x^2 - a^2 = (x - a)(x + a)$$

$$x^3 - a^3 = (x - a)(x^2 + a\,x + a^2)$$

$$x^4 - a^4 = (x - a)(x^3 + a\,x^2 + a^2\,x + a^3)$$

$$x^5 - a^5 = (x - a)(x^4 + a\,x^3 + a^2\,x^2 + a^3\,x + a^4)$$

We can see that $x^6 - a^6 = (x-a)(x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5)$

$$x^n - a^n = (x-a)(x^{n-1} + a^1x^{n-2} + \ldots a^{n-1})$$

Now, if $f(x) = x^n$, $f'(x) = \lim_{h \to 0} \dfrac{(x+h)^n - x^n}{h}$

$$= \lim_{h\to 0} \frac{\left[(x + h) - x\right]\left[(x + h)^{n-1} + (x + h)^{n-2}\,x + \ldots + (x + h)\,x^{n-2} + x^{n-1}\right]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left[(x + h)^{n-1} + (x + h)^{n-2}\,x + \ldots + (x + h)\,x^{n-2} + x^{n-1}\right]}{\cancel{h}}$$

$$= \lim_{h\to 0} (x + h)^{n-1} + (x + h)^{n-2}\,x + \ldots + (x + h)\,x^{n-2} + x^{n-1}$$

$$= (x + 0)^{n-1} + (x + 0)^{n-2}\,x + \ldots + (x + 0)\,x^{n-2} + x^{n-1}$$

$= x^{n-1} + x^{n-2}\cdot x + \ldots + x \cdot x^{n-2} + x^{n-1}$

$= x^{n-1} + x^{n-1} + \ldots + x^{n-1} + x^{n-1}$

$= n\,(x)^{n-1}$

$$\boxed{\text{if } f(x) = x^n \quad \text{then} \quad f'(x) = n \cdot x^{n-1}}$$

Thus $\frac{d}{dx}x^5 = \underline{5\cdot x^4}$ and $\frac{d}{dx}x^{27} = \underline{27\cdot x^{26}}$

$f(x) = x^{10}$

$f'(x) = 10\cdot x^9$

Although we haven't yet proved this rule for negative, rational or irrational powers we will ultimately prove it to be true for all reals.

Example: Determine y' if

a) $y = \frac{1}{x^3} = x^{-3}$

$= -3x^{-4}$

$f'(x) = -3\cdot\frac{1}{x^4}$

b) $y = \sqrt{x} = x^{\frac{1}{2}}$

$y' = \frac{1}{2}x^{-\frac{1}{2}}$

$y' = \frac{1}{2}\frac{1}{\sqrt{x}}$

c) $y = x^{-\frac{2}{3}}$

$y' = -\frac{2}{3}x^{-5/3}$

Note also that the power rule is used to differentiate a power of x and not functions like $y = (x+1)^3$ or $y = (5x^2+3)^4$

$y = x^3 + 1$

$y' = \frac{d(x^3)}{dx} + \frac{d(1)}{dx}$

$y' = 3x^2 + 0$

The Product Rule for Derivatives

If f and g are differentiable at x, then so is the product $f \bullet g$, and

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

or $$\frac{d}{dx}\left[f\bullet g\right] = f'\bullet g + f\bullet g'$$

eg $y = (3x^2)\cdot(5x^2)$

Proof: $$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

If we add and subtract $f(x+h)g(x)$ in the numerator, we then obtain

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}g(x)\cdot\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$f(x)$ $g(x)$

Examples

1. Determine the derivative of the $y=\underbrace{(x^2+3)}_{f(x)}\underbrace{(2x-5)}_{g(x)}$ by

a) using the product rule

$y' = f(x)\cdot g'(x) + g(x)\cdot f'(x)$

$= (x^2+3)(2) + (2x-5)(2x)$

$= 2x^2+6 + 4x^2-10x$

$y' = 6x^2-10x+6$

b) expanding and then differentiating

$y = (x^2+3)(2x-5)$

$y = 2x^3+6x-5x^2-15$

$y' = 6x^2+6-10x+0$

same thing!

2. Given the following: $h=f\bullet g$, $f(4)=3$, $f'(4)=-2$, $g(4)=-7$ and $g'(4)=-5$, determine $h'(4)$.

$h = f\cdot g$

$\boxed{h' = f\cdot g' + g\cdot f'}$

$h'(4) = (3)(-5) + (-7)(-2)$

$= -15 + 14$

$h'(4) = -1$

3. Determine $\dfrac{dy}{dx}$ if $y=f(g\bullet h)$

$y' = f(\underbrace{g\cdot h})' + (g\cdot h)f'$

$= f[g\cdot h' + h\cdot g'] + (g\cdot h)\cdot f'$

The Quotient Rule for Derivatives

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{df(x)}{dx}\bullet g(x)-\frac{dg(x)}{dx}\bullet f(x)}{g^2(x)} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{f}{g}\right)=\frac{f'g-g'f}{g^2}$$

Note that order for this rule is very important.

Proof: Let $h=\dfrac{f}{g}$, then $f=$ $h\cdot g$.

Using the product rule, we then have $f'=h'g+g'h$

$f'-g'\cdot h = h'\cdot g$

$\div g \qquad \div g$

Solving for h' we get $h'=\dfrac{f'-g'\cdot h}{g}$, and then by substituting for h we obtain

$h'=\dfrac{f'-g'\left(\frac{f}{g}\right)}{g} \rightarrow \dfrac{\frac{f'g}{g}-\frac{g'f}{g}}{g}$

Converting to a common denominator yields $h'=\dfrac{f'g-g'f}{g^2}$

Examples

1. Determine the derivative of $y=\frac{1}{x^5}$ by $f(x)$ $g(x)$

a) Using the quotient rule	b) rewriting as a negative power and then using the power rule
$y' = \frac{f'g - g'f}{g^2}$ $= \frac{0(x^5)-(5x^4)(1)}{x^{10}}$ $= \frac{-5x^4}{x^{10}} = -5x^{-6}$	$y=\frac{1}{x^5} = x^{-5}$ $y' = -5x^{-6}$

2. Determine the equation of the tangent to $y=\frac{5x+1}{3x-1}$ at $x=1$ $f(x)$ $g(x)$

$$y' = \frac{f'g-g'f}{g^2}$$

$$= \left.\frac{5(3x-1)-(3)(5x+1)}{(3x-1)^2}\right|_{x=1} = \frac{(10)-(18)}{4} = \underline{\underline{-2}}$$

3. What is $\lim_{h\to 0}\frac{\frac{8}{(x+h)^{12}}-\frac{8}{x^{12}}}{h}$ $= \frac{d}{dx}\left(\frac{8}{x^{12}}\right) = \frac{d}{dx}\left(8x^{-12}\right)$

$$= 8(-12)\cdot x^{-13}$$

$$= -96x^{-13}$$

Higher Order Derivatives

The process of finding the derivative of a function can also be applied to the derivative itself. (ie we can find the derivative of the derivative). y' or $\frac{dy}{dx}$ is called the first derivative of y with respect to x.

The second derivative with respect to x is $y''=\frac{dy'}{dx}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$.

The third derivative is $y'''=\frac{dy''}{dx}=\frac{d^3y}{dx^3}$ and the n^{th} derivative of y with respect to x is $y^{(n)}=\frac{d^ny}{dx^n}$

Example: If $y=x^3+5x^2-7x-18$, what is $\frac{d^2y}{dx^2}$?