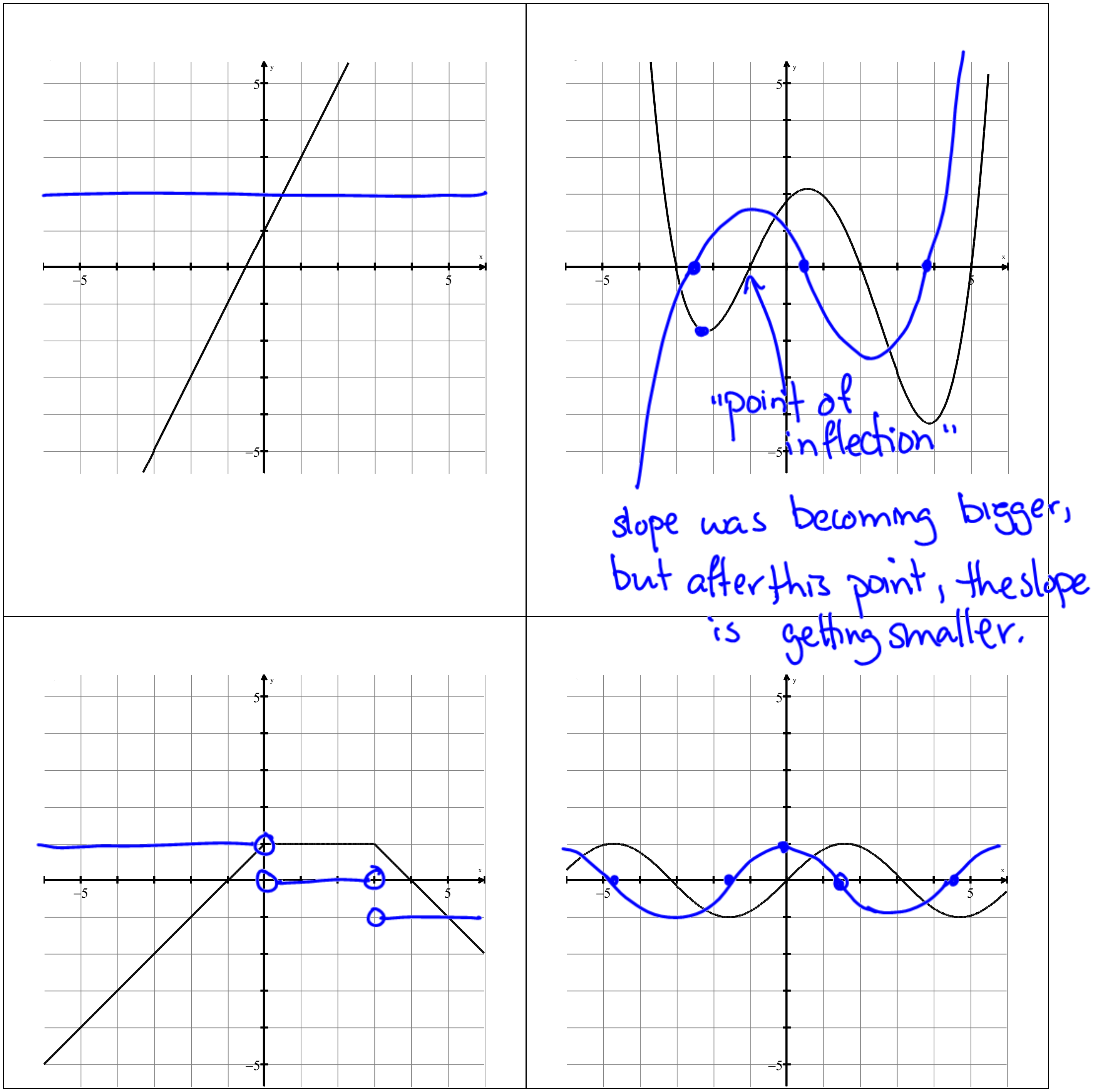
2.3 Warmup

Graph the derivative of the following functions. Where necessary, approximate the derivative.

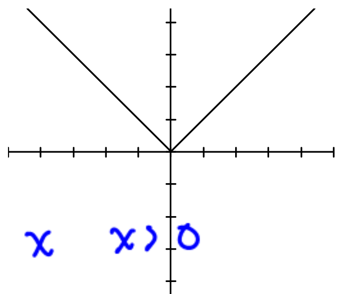


Differentiability

Must a function have a derivative at each point where the function is defined?
Or
If $f(a)$ is defined, must $f'(a)$ be defined?

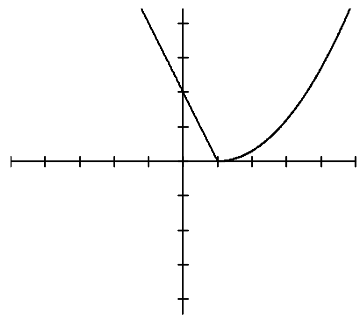
Places where $f'(x)$ ***might fail to exist.*** - places where you can't find a derivative.

1. Corner - left hand derivative $\neq$ right hand derivative.



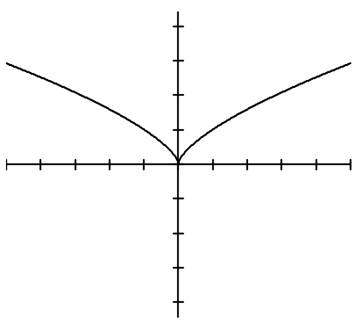
$$|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

$y = |x|$

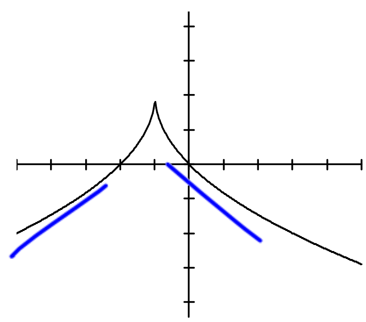


$$y = \begin{cases} -(x-1) & x \le 1 \\ (x-1)^2 & x > 1 \end{cases}$$

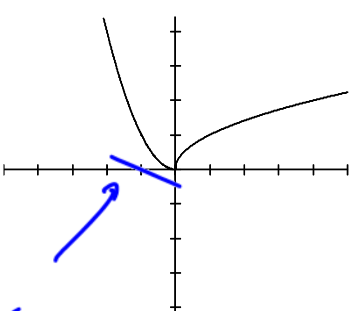
2. Cusp - one derivative is ∞, the other $-\infty$



$y = x^{\frac{2}{3}}$



$y = -\sqrt{|x+1|} + 2$



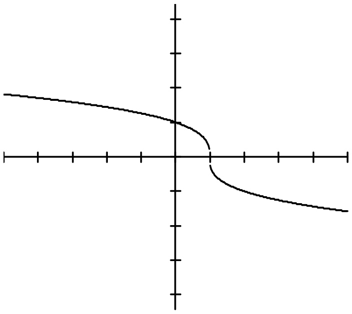
Corner or cusp?

slope is approaching 0

3. Vertical tangent

m_{tan} exists and left and right tangents are equal

but m_{tan} = vertical line

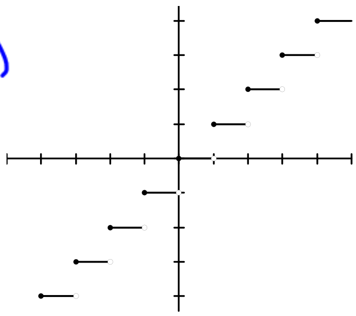


$y = -\sqrt[3]{x+1}$

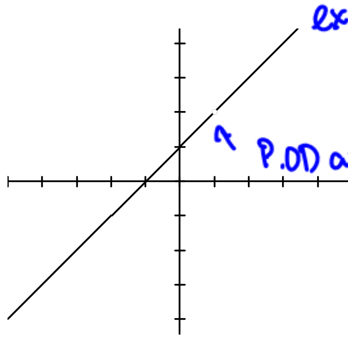
4. Points of discontinuity

differentiable everywhere except $x=1$

$$\frac{f(x+h)-f(x)}{h}$$



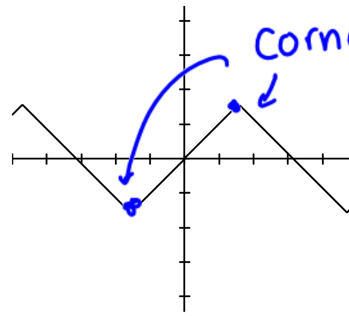
$y = \text{int}(x)$



P.OD at (1,2)

$y = \dfrac{x^2-1}{x-1}$

corners

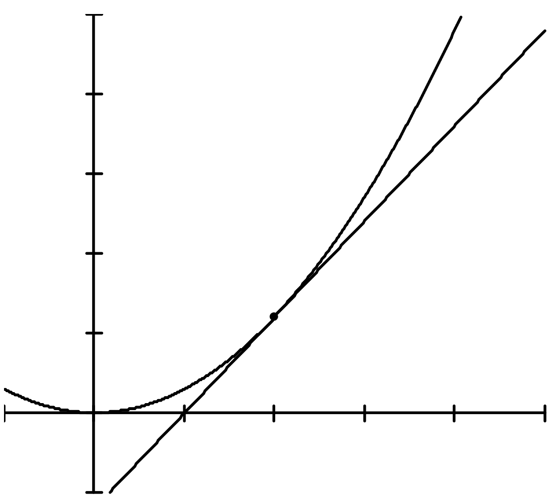


$y = \sin^{-1}(\sin x)$

* If a function has a derivative at a, then f must be ___differentiable___ at a.

Local Linearity

If a function is differentiable at point a, then it is ___locally___ ___linear___ at point a. In other words, the graph of the function resembles the ___tangent___ ___line___ in a small interval around a.



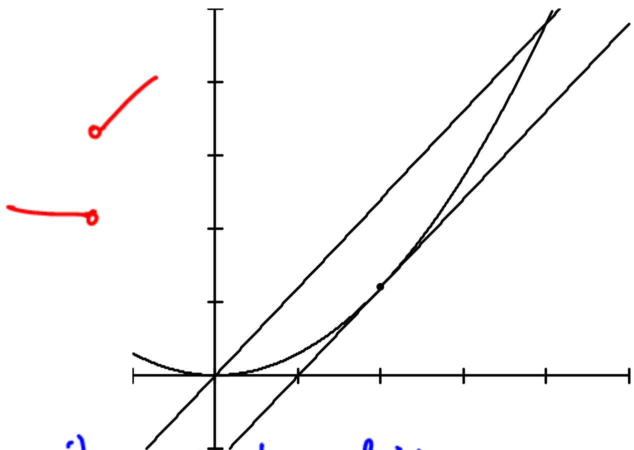
zooming in reveals that the corner is really a corner.

Compare the graphs of $y = |x| + 1$ and $y = \sqrt{x^2 + 0.0001} + 0.99$ at $x = 1$. Do either of these functions appear to be differentiable at $x = 0$?

differentiable, when you zoom in, the corner turns out to be a curve.

Derivatives on a Calculator

How does your calculator calculate the derivative of a function at point *a*?



Your calculator comes up with a **numerical estimate** for the derivative at a point using a **symmetric difference quotient**.

math > 8:nDeriv (function, x, 3)

$$\frac{d}{d\boxed{x}}(\square)\Big|_{\boxed{x} = \square}$$

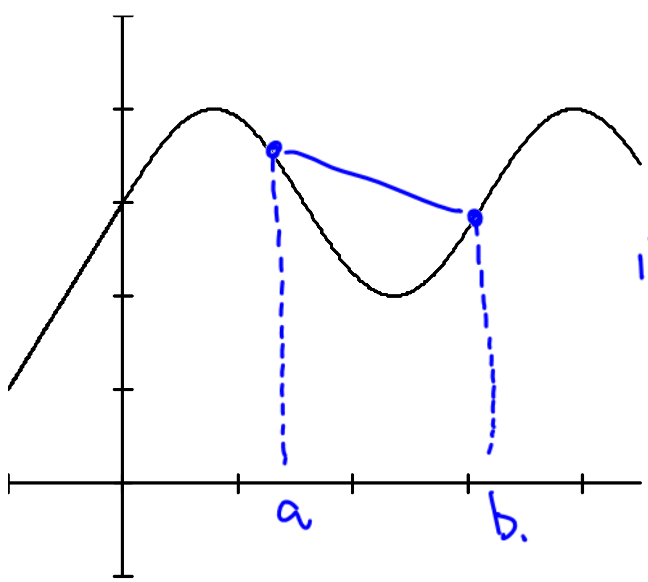
it actually finds msec near the x value.

$$\frac{d(x^2)}{dx}\Big|_{x=3}$$

nDeriv$(x^2, x, 3)$

Intermediate Value Theorem for Derivatives

If *a* and *b* are two points on an interval where *f* is differentiable, then $f'(x)$ must take on every value between $f'(a)$ and $f'(b)$ on the interval $[a, b]$



ivt: there exists a coordinate with a y-value between f(a) and f(b).

ivt for derivatives: there must exist a place where f'(x) is between f'(a) and f'(b)

Warmup 2.4

1. Use your calculator to determine the derivative of $y = \left|16 - x^2\right|$ at $x = 4$. Is this a reasonable value for the derivative at this point? Why or why not?

2. Graph the functions $f(x) = x^2$, $g(x) = \text{nDeriv}\left(x^2, x, x\right)$ and $h(x) = \left(f(x+.001) - f(x-.001)\right)/.002$.
   What function does the graph of $g(x)$ appear to represent? Does this make sense?

   How does the graph of $h(x)$ compare to the graph of $g(x)$?

3. Predict the answers to the following:

a) $\dfrac{d}{dx}5$ b) $\dfrac{d}{dx}x$ c) $\dfrac{d}{dx}x^2$

d) $\dfrac{d}{dx}x^3$ e) $\dfrac{d}{dx}x^4$ f) $\dfrac{d}{dx}x^{27}$

4. Use limits to verify your answer to “3e”.

Rules for Differentiation

Derivative of a Constant Function

If $f(x) = k$ then ___________

Power Rule for Positive Integer Powers

If $f(x) = x^n$ then ____________

Constant Multiple of a Function

If $f(x) = k \bullet g(x)$ then ___________

Sum and Difference Rule

$$\frac{d}{dx}\left(f(x) \pm g(x) \right) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Product Rule

$$\frac{d}{dx}\left(f(x) \bullet g(x) \right) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)} \right) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

The Power Rule for Derivatives

If $y = x^n$ then $y' = n\,x^{n-1}$ for all real values of n

This can be proved for all positive integers n in the following way

Proof : The factor theorem says: If a is a root of $f(x) = 0$ then $x - a$ is a factor of $f(x)$

If $f(x) = x^n - a^n$ then $f(a) =$___ and consequently __________ is a _____________ of _______

By observing the pattern

$$x - a = x - a$$
$$x^2 - a^2 = (x - a)(x + a)$$
$$x^3 - a^3 = (x - a)\left(x^2 + a\,x + a^2\right)$$
$$x^4 - a^4 = (x - a)\left(x^3 + a\,x^2 + a^2\,x + a^3\right)$$
$$x^5 - a^5 = (x - a)\left(x^4 + a\,x^3 + a^2\,x^2 + a^3\,x + a^4\right)$$

We can see that

$x^n - a^n =$____

Now, if $f(x) = x^n$, $f'(x) =$________________

$$= \lim_{h\to 0} \frac{\left[(x + h) - x\right]\left[(x + h)^{n-1} + (x + h)^{n-2}\,x + \;\ldots\; + (x + h)\,x^{n-2} + x^{n-1}\right]}{h}$$

$$= \lim_{h\to 0} \frac{h\,\left[(x + h)^{n-1} + (x + h)^{n-2}\,x + \;\ldots\; + (x + h)\,x^{n-2} + x^{n-1}\right]}{h}$$

$$= \lim_{h\to 0} (x + h)^{n-1} + (x + h)^{n-2}\,x + \;\ldots\; + (x + h)\,x^{n-2} + x^{n-1}$$

$$= (x + 0)^{n-1} + (x + 0)^{n-2}\,x + \;\ldots\; + (x + 0)\,x^{n-2} + x^{n-1}$$

= ____

= _______________

Thus $\dfrac{d}{dx}x^5 =$ ______ and $\dfrac{d}{dx}x^{27} =$ ______

Although we haven't yet proved this rule for negative, rational or irrational powers we will ultimately prove it to be true for all reals.

Example: Determine y' if

a) $y = \dfrac{1}{x^3}$

b) $y = \sqrt{x}$

c) $y = x^{-\frac{2}{3}}$

Note also that the power rule is used to differentiate a power of x and not functions like $y = (x+1)^3$ or $y = (5x^2 + 3)^4$

The Product Rule for Derivatives

If f and g are differentiable at x, then so is the product $f \bullet g$, and

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

or $\dfrac{d}{dx}\left[f \bullet g\right] = f' \bullet g + f \bullet g'$

Proof: $\dfrac{d}{dx}\left[f(x)\,g(x)\right] = \lim\limits_{h \to 0} \dfrac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$

If we add and subtract $f(x+h)\,g(x)$ in the numerator, we then obtain

$$\frac{d}{dx}\left[f(x)\,g(x)\right] = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

$$= \lim_{h \to 0}\left[f(x+h)\,\frac{g(x+h) - g(x)}{h} + g(x)\,\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Examples

1. Determine the derivative of the $y=\left(x^2+3\right)(2x-5)$ by

a) using the product rule

b) expanding and then differentiating

2. Given the following: $h=f\bullet g$, $f(4)=3$, $f'(4)=-2$, $g(4)=-7$ and $g'(4)=-5$, determine $h'(4)$.

3. Determine $\frac{dy}{dx}$ if $y=f\bullet g\bullet h$

The Quotient Rule for Derivatives

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{df(x)}{dx}\bullet g(x)-\frac{dg(x)}{dx}\bullet f(x)}{g^2(x)} \quad \text{or} \quad \frac{d}{dx}\left(\frac{f}{g}\right)=\frac{f'g-g'f}{g^2}$$

Note that order for this rule is very important.

Proof: Let $h=\frac{f}{g}$, then $f=$ ______

Using the product rule, we then have $f'=h'g+g'h$

Solving for h' we get $h'=$ ________________, and then by substituting for h we obtain $h'=$ ____________

Converting to a common denominator yields $h'=$ ________________

Examples

1. Determine the derivative of $y = \frac{1}{x^5}$ by

a) Using the quotient rule	b) rewriting as a negative power and then using the power rule

2. Determine the equation of the tangent to $y = \frac{5x+1}{3x-1}$ at $x = 1$

3. What is $\lim_{h \to 0} \frac{\frac{8}{(x+h)^{12}} - \frac{8}{x^{12}}}{h}$

Higher Order Derivatives

The process of finding the derivative of a function can also be applied to the derivative itself. (ie we can find the derivative of the derivative). y' or $\frac{dy}{dx}$ is called the first derivative of *y* with respect to *x*.

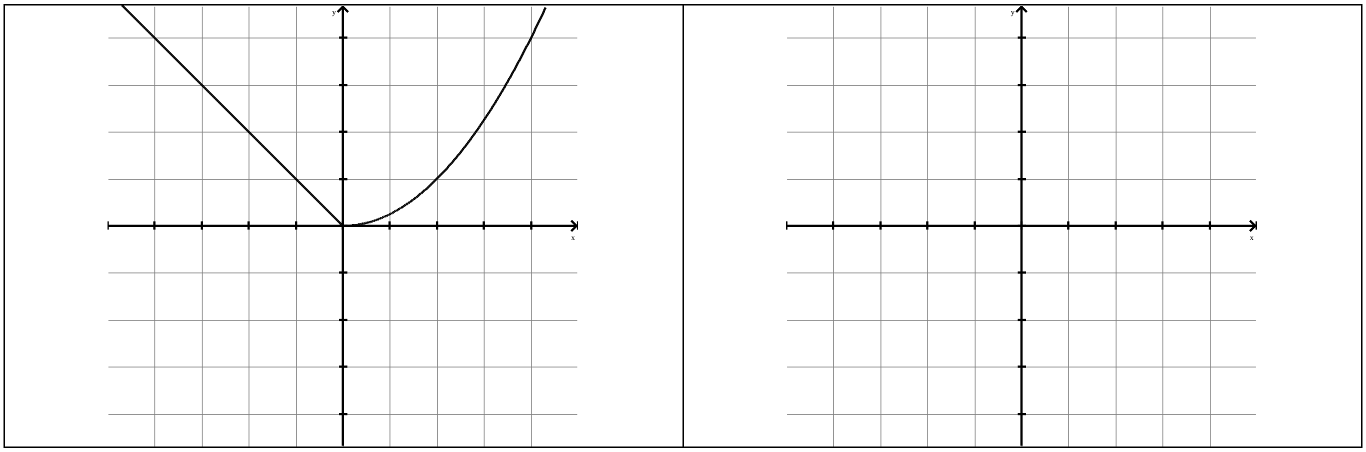
The second derivative with respect to *x* is $y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

The third derivative is $y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$ and the n^{th} derivative of *y* with respect to *x* is $y^{(n)} = \frac{d^ny}{dx^n}$

Example: If $y = x^3 + 5x^2 - 7x - 18$, what is $\frac{d^2y}{dx^2}$?

Warmup 2.5

1. Sketch a graph of the derivative of the following function



2. Find the derivatives of the following functions

a) $y = (3x - 4)(x^2 - 7x + 3)$

b) $f(x) = \dfrac{4x^3}{x^2 + 3}$

c) $y = \dfrac{1}{f(x)}$

d) $y = \sqrt{x}$

3. Suppose g and h are differentiable functions at $x = 0$ and that $g(0) = 5,\ g'(0) = -2$
$h(0) = -3,\ h'(0) = 4$. Find the value of the following derivatives at $x = 0$

a) $\dfrac{d}{dx}(g\,h)$

b) $\dfrac{d}{dx}(5\,g - 9\,h)$

c) $\dfrac{d}{dx}\left(\dfrac{g}{h}\right)$

Rules for Differentiation - Tangent Lines

6. Find the equation of the tangent line to the curve $y = 2x^2 - 6x + 7$ when $x = 2$.

7. Find all points on the curve $y = x^3 + 2x^2 + x - 7$ where the tangent line is horizontal.

8. Find all points on the curve $y = \frac{1}{x}$ where the slope of the normal is 4.

9. Find the tangent to the curve $y = \frac{8}{4 + x^2}$ at the point (2 , 1)

10. If gas in a cylinder is to be maintained at a constant temperature T, the pressure P is related to the volume V by the formula:

$$P = \frac{nRT}{V - nB} - \frac{an^2}{V^2}$$

where a, b, n, and R are constants. Find $\frac{dP}{dV}$ and describe what it means.

2.6 Warmup

1. Determine $\frac{d}{dx}\left(\frac{fg}{h}\right)$

2. Determine $\frac{d}{dx} f^3$ (Hint: Treat f^3 as $f \bullet f \bullet f$ and use the product rule)

3. Determine $\frac{d}{dt}\left(\frac{At-1}{A^2t}\right)$ where A is a constant.

4. If $y = x^n$ what is $y^{(n)}$ or $\frac{d^n y}{dx^n}$ (or what is the n^{th} derivative)?

5. If you keep on differentiating a polynomial function, will the derivative eventually become zero? Why or why not?