Warmup 1.12

1. Find the instantaneous rate of change of the volume of a sphere with respect to the radius. The formula for volume is $V = \frac{4\pi r^3}{3}$.

$$m_{tan} = \lim_{h\to 0} \frac{\left(\frac{4\pi}{3}(r+h)^3 - \frac{4\pi}{3}(r^3)\right)}{h}$$

$$= \lim_{h\to 0} \left(\frac{4\pi}{3}\right)\left(\frac{1}{h}\right)\left[(r+h)^3 - r^3\right]$$

$$= \lim_{h\to 0} \left(\frac{4\pi}{3}\right)\left(\frac{1}{h}\right)\left(\left[r^3 + 3r^2h + 3rh^2 + h^3\right] - r^3\right)$$

$$= \lim_{h\to 0} \left(\frac{4\pi}{3}\right)\left(\frac{1}{h}\right)\left(3r^2h + 3rh^2 + h^3\right)$$

$$= \frac{4\pi}{3}(3r^2) \quad \text{or} \quad \boxed{m_{tan} = 4\pi r^2}$$

2. Find all points on the curve $y = \frac{1}{x-5}$ where the slope of the tangent is $-\frac{1}{4}$.

$$m_{tan} = \lim_{h\to 0} \frac{\frac{1}{a+h-5} - \frac{1}{a-5}}{h} = \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{(a-5) - (a+h-5)}{(a+h-5)(a-5)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(a+h-5)(a-5)}\right)$$

$$m_{tan} = \frac{-1}{(a-5)^2}$$

$$\frac{-1}{4} = \frac{-1}{(a-5)^2}$$

$$(a-5)^2 = 4$$

$$a-5 = \pm 2$$

$$a = \pm 2 + 5$$

$a = 7$ or $a = 3$

$f(7) = \frac{1}{2}$ $\quad f(3) = -\frac{1}{2}$

$\left(7, \frac{1}{2}\right)$ and $\left(3, -\frac{1}{2}\right)$