Rates of Change

Mr. O. decides to go on a walking program. During the second week of the program, he walks a total of 5 km. During the seventh, his total is 23 km. What was the average rate of change of total distance walked during that time?

$\frac{18}{5} = 3.6\text{ km}$

(w, d)

$(2, 5)$

$(7, 23)$

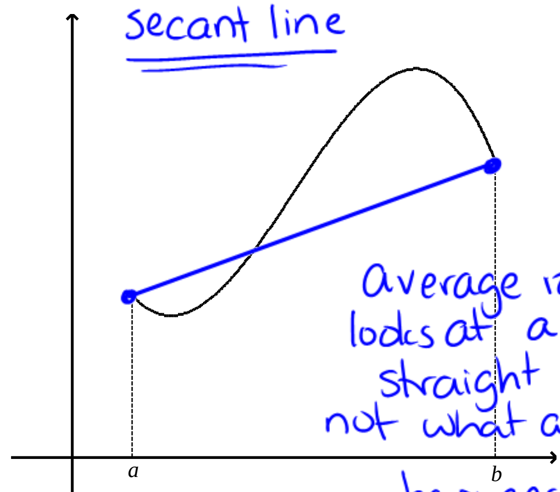
Average Rate of Change of a Function

The **average rate of change** of a function over an interval is defined to be the amount of change in the function divided by the length of the interval.

secant line

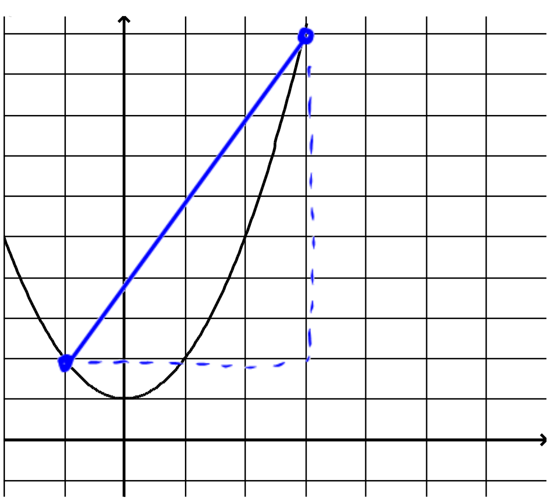
Average rate of change on [*a*,*b*] = $\dfrac{\Delta f(x)}{\Delta x}$

$$= \frac{f(b) - f(a)}{b - a}$$



average rate looks at a straight line, not what actually happens.

Example 1: Determine the average rate of change of the function $y = x^2 + 1$ on $[-1, 3]$



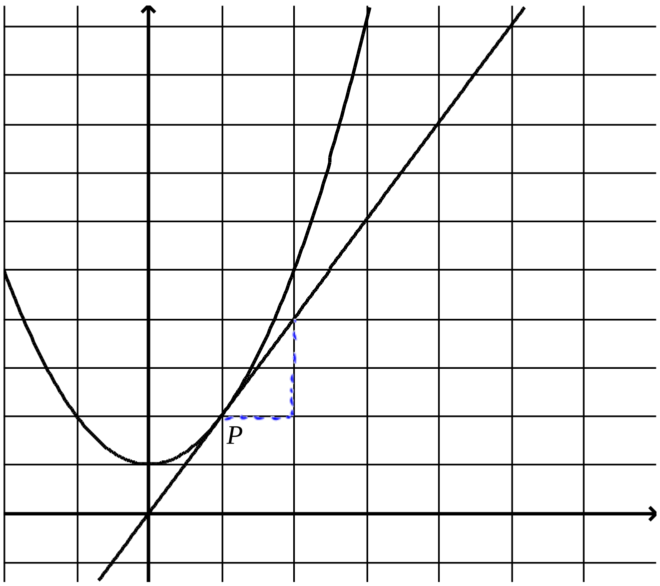
$m_{sec} = \frac{2}{1}$ (from graph)

or

$m_{sec} = \dfrac{f(3) - f(-1)}{3 - (-1)} = \dfrac{10 - 2}{4}$

$= \frac{8}{4}$ or $\frac{2}{1}$

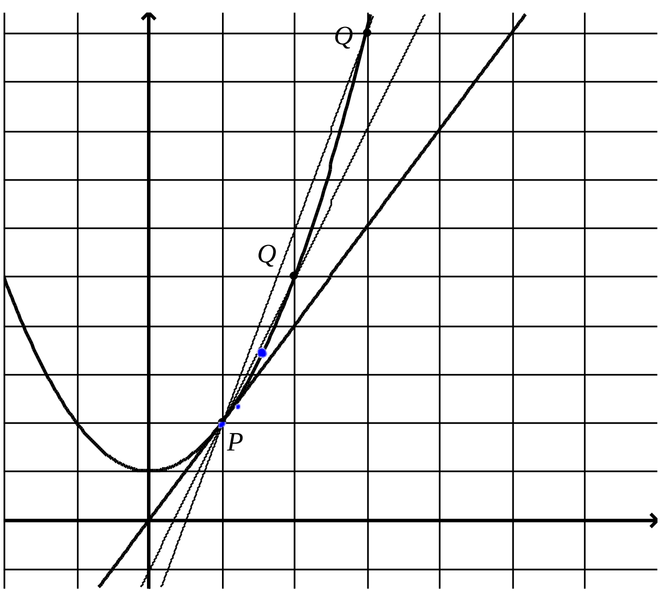
Suppose now that rather than finding the average rate of change, we would like to find out what is the rate of change at a specific point. This is known as the ***instantaneous rate of change***. For the above function, for instance, we might like to know what is the rate of change when x=1. How can we determine this?



- you could estimate the slope
- draw the line
- find the slope of this line.

"tangent line"

To find the instantaneous rate of change at $x = 1$, we then need to find the slope of the tangent at that point



Notice that as Q moves along the curve to P, the slopes of the secants get closer to the slope of the tangent.

P	Q	Slope of secant
(1, 2)	~~(3,9)~~ (3,10)	$\frac{8}{2} = 4$
(1, 2)	(2, 5)	$\frac{3}{1} = 3$
(1, 2)	(1.5, 3.25)	$\frac{1.25}{.5} = 2.5$
(1, 2)	(1.1, 2.21)	$\frac{.21}{.1} = 2.1$
(1, 2)	(1.05, 2.1025)	$\frac{.1025}{.05} = 2.05$

To find the slope of the tangent (instantaneous rate of change) at a point, you

a) Choose a point Q "close" to P, and calculate the slope of the secant joining P and Q.
b) Find the limiting value of the secant slope as Q approaches P along the curve.
c) The ***slope of the curve at P*** is defined to be this value, and the ***tangent to the curve at P*** is defined to be the line with this slope passing through P.

$m_{sec} = \frac{f(b) - f(a)}{b-a}$ as b and a get very close together $m_{sec} \to m_{tan}$

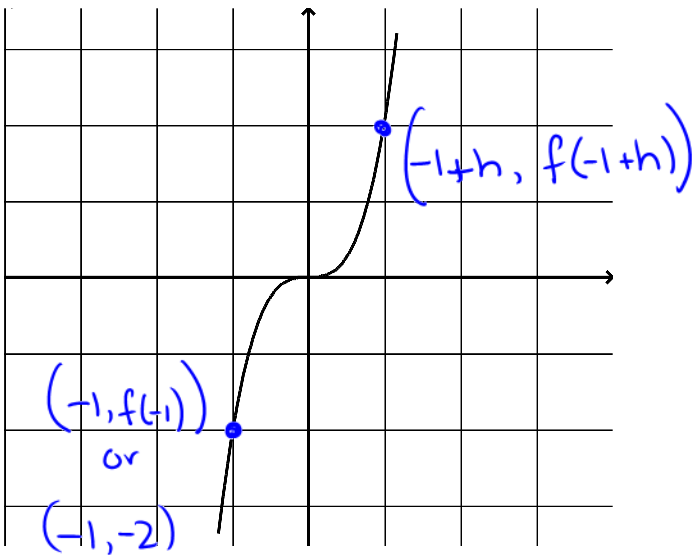
Thus, we have the following

Instantaneous rate of change at a point = slope of the curve at that point

=slope of the tangent at that point

=the limit of the slope of the secant

Example 2. Find the slope of the tangent to the curve $y = 2x^3$ at $x = -1$.



$$m_{tan} = \lim_{h \to 0} m_{sec}$$

use

$$m_{sec} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(-1+h) - f(-1)}{(-1+h) - (-1)}$$

$$= \frac{2(-1+h)^3 - (-2)}{h}$$

$$\boxed{(h-1)^3 = 1h^3 - 3h^2 + 3h - 1}$$

$$m_{tan} = \lim_{h \to 0} \frac{2(-1+h)^3 - (-2)}{h}$$

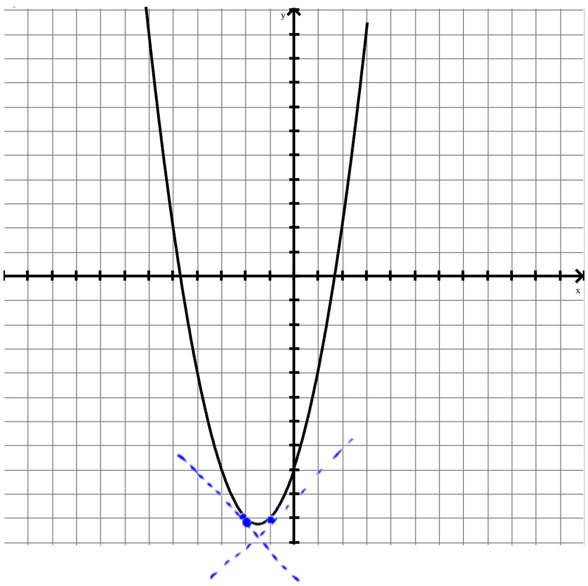
$$= \lim_{h \to 0} \frac{2(h^3 - 3h^2 + 3h - 1) + 2}{h}$$

$$= \lim_{h \to 0} \frac{2h^3 - 6h^2 + 6h}{h}$$

$$= \lim_{h \to 0} 2h^2 - 6h + 6$$

$$m_{tan} = 6$$

Example 3. Find the slope of the curve $y = x^2 + 3x - 8$ at $x = -2$



$$m_{sec} = \frac{f(b)-f(a)}{b-a}$$

$$m_{tan} = \lim_{h \to 0} \frac{f(-2+h)-f(-2)}{-2+h-(-2)}$$

$$= \lim_{h \to 0} \frac{\left[(-2+h)^2+3(-2+h)-8\right]-\left[(-2)^2+3(-2)-8\right]}{h}$$

$$= \lim_{h \to 0} \frac{(h^2-4h+4+-6+3h-8)-(-10)}{h}$$

$$= \lim_{h \to 0} \frac{h^2-h}{h}$$

$$= \lim_{h \to 0} h-1$$

$$m_{tan} = -1$$

How would you expect the slope of the curve at $x = -1$ to compare with the slope at $x = -2$?

positive, but the same absolute value, because the graph is symmetrical

Does the slope of the curve depend upon where you are on the curve?

yes

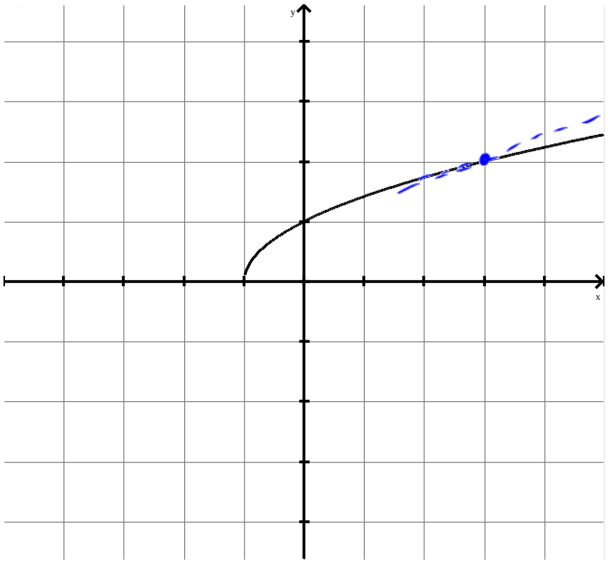
What appears to happen to the slopes as x becomes increasingly large positively?

increasingly large $\rightarrow$ $\displaystyle m_{tan} = \infty$ ($x \to \infty$)

What would you expect the slope of the curve to be at the vertex?

$m_{tan} = 0$

Example 4. Find the instantaneous rate of change of the function $y=\sqrt{x+1}$ at $x=3$.



$$m_{tan} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{3+h-3}$$

$$\lim_{h\to 0} \frac{\sqrt{(3+h)+1}-\sqrt{(3+1)}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

$$= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}^{\,1}}{\cancel{h}(\sqrt{4+h}+2)}$$

$$m_{tan} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Find the slope of the curve above at $x=a$. (ie. at any point, rather than a specific point)

$$m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{a+h-a}$$

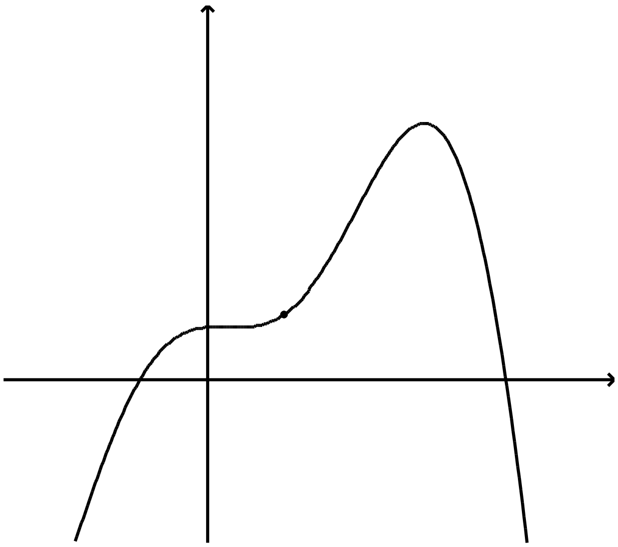
$$= \lim_{h\to 0} \frac{\sqrt{a+h+1}-\sqrt{a+1}}{h} \cdot \frac{\sqrt{a+h+1}+\sqrt{a+1}}{\sqrt{a+h+1}+\sqrt{a+1}}$$

$$= \lim_{h\to 0} \frac{(a+h+1)-(a+1)}{h(\sqrt{a+h+1}+\sqrt{a+1})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{a+h+1}+\sqrt{a+1})}$$

$$m_{tan} = \frac{1}{2\sqrt{a+1}}$$

Slope of a Curve at Any Point

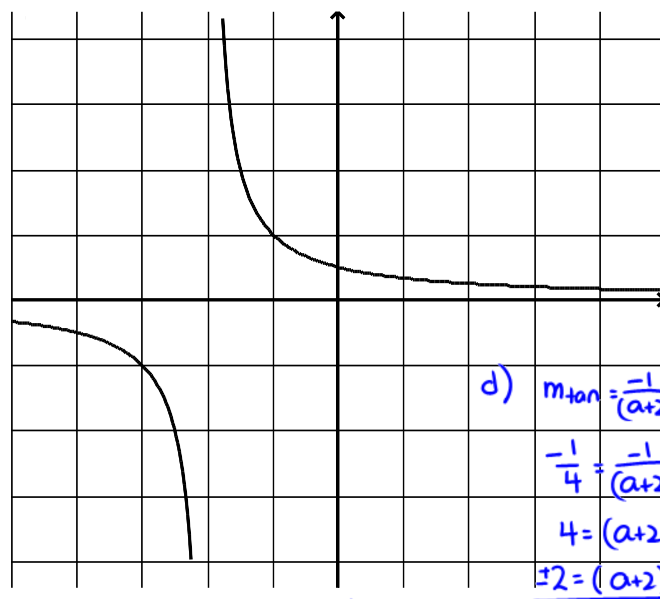


The slope of the secant = $\dfrac{f(a+h)-f(a)}{a+h-a}$

> The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is *
>
> $$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

HW #1-8 in workbook
p87 #1-21 odd

Example 5: Let $f(x) = \dfrac{1}{x+2}$



a) Find the slope of the curve at $x = a$

b) Find the slope of the curve at $x = 1$

c) Find the equation of the tangent at $x = 1$

d) For what value(s) of x does the slope equal $-\dfrac{1}{4}$?

a) $m_{tan} = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

$= \lim_{h \to 0} \left[\dfrac{1}{a+h+2} - \dfrac{1}{a+2}\right] \cdot \dfrac{1}{h}$

$= \lim_{h \to 0} \dfrac{(a+2) - (a+h+2)}{(a+h+2)(a+2)} \cdot \dfrac{1}{h}$

$= \lim_{h \to 0} \dfrac{-\cancel{h}}{(a+h+2)(a+2)} \cdot \dfrac{1}{\cancel{h}} = \dfrac{-1}{(a+2)^2}$

b) $m_{tan} = \dfrac{-1}{(a+2)^2}$

m_{tan} at $x=1$ $= \dfrac{-1}{(1+2)^2} = \dfrac{-1}{9}$

c) when $x=1$ $y = f(1)$

$= \dfrac{1}{(1+2)}$

$f(1) = \dfrac{1}{3}$

y-coord

$y - \frac{1}{3} = \frac{-1}{9}(x-1)$

slope

x-coord

d) $m_{tan} = \dfrac{-1}{(a+2)^2}$

$\dfrac{-1}{4} = \dfrac{-1}{(a+2)^2}$

$4 = (a+2)^2$

$\pm 2 = (a+2)$

$\boxed{a = 0 \text{ or } -4}$