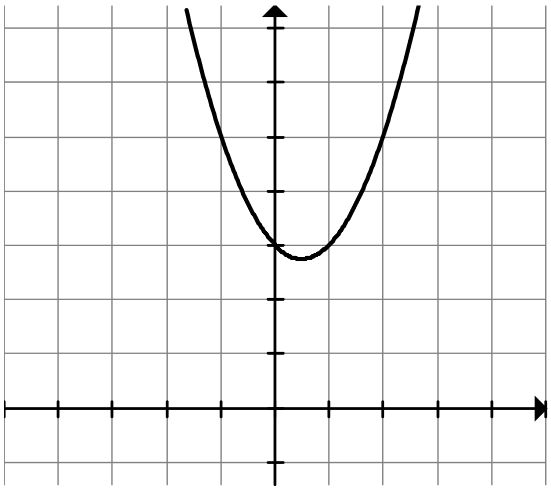
Definition of the Limit of a Function

Calculus 12
Unit 1.2

If $f(x) = x^2 - x + 3$, what happens to the values of $f(x)$ as x gets closer to 2?

Graphical Analysis



Numerical Analysis

x	$f(x)$	x	$f(x)$
1.0	3	3.0	9
1.5	3.75	2.5	6.75
1.9	4.71	2.1	5.31
1.99	4.97	2.01	5.03
1.999	4.997	2.001	5.003

It appears that the values of $f(x)$ get closer and closer to 5 as x gets closer to 2. This is an example of the idea of a limit and is denoted as $\lim_{x\to 2} f(x) =$ 5. (This is read as "the limit of $f(x)$ as x approaches 2.")

Thus $\lim_{x\to a} f(x)$ means to describe what happens to the values of $f(x)$ as x gets closer and closer to a.

Definition $\qquad \lim_{x\to a} f(x) = L$

The limit of $f(x)$, as x approaches a, equals L, if we can make the values of $f(x)$ arbitrarily close to L (as close as we like) by taking x sufficiently close to a but **not** equal to a.

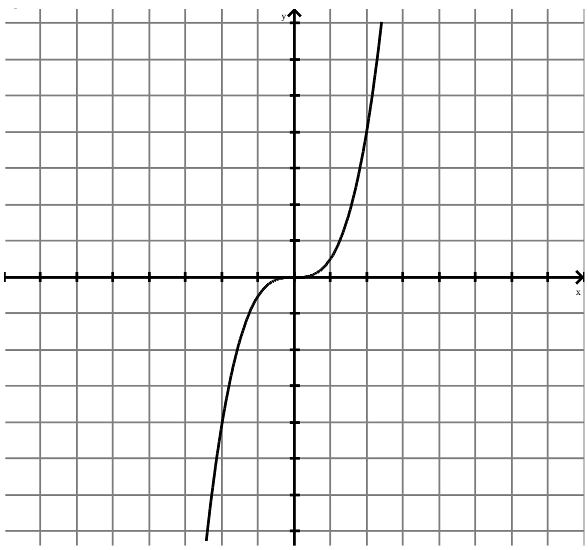
Precise Definition of the Limit

The function f has limit L as x approaches a, if for any given ε, there is a positive number δ such that for all x

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

One and Two sided limits

Consider the function $f\left(x\right)=\frac{1}{2}x^{3}$



What happens to the values of $f\left(x\right)$ as you approach 2 from the left hand side?

$\lim\limits_{x\to 2^{-}} f\left(x\right)=$ 4 "left hand limit"

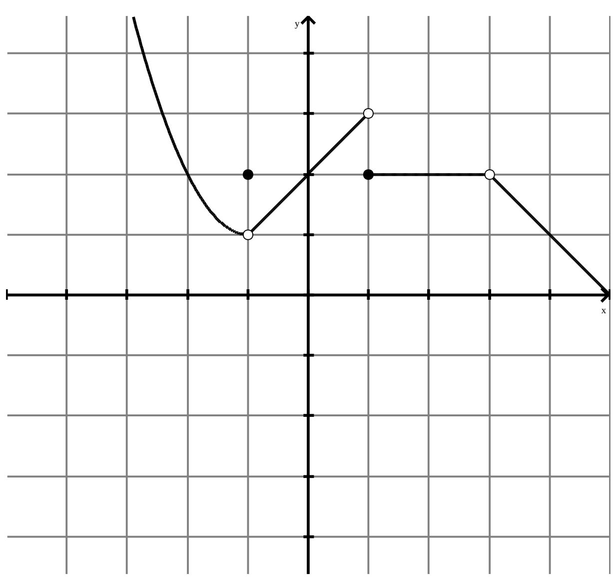
What happens to the values of $f\left(x\right)$ as you approach 2 from the right hand side?

$\lim\limits_{x\to 2^{+}} f\left(x\right)=$ 4 "right hand limit"

Thus $\lim\limits_{x\to 2} f\left(x\right)=$ 4

this limit only exists if left hand and right hand limits are equal.

Use the graph of $y=f\left(x\right)$ below to determine each of the limits:



$\lim\limits_{x\to -2^{-}} f\left(x\right)$ 2	$\lim\limits_{x\to -2^{+}} f\left(x\right)$ 2	$\lim\limits_{x\to -2} f\left(x\right)$ 2
$\lim\limits_{x\to -1^{-}} f\left(x\right)$ 1	$\lim\limits_{x\to -1^{+}} f\left(x\right)$ 1	$\lim\limits_{x\to -1} f\left(x\right)$ 1 even though f(-1) = 2
$\lim\limits_{x\to 1^{-}} f\left(x\right)$ 3	$\lim\limits_{x\to 1^{+}} f\left(x\right)$ 2	$\lim\limits_{x\to 1} f\left(x\right)$ no limit exists.
$\lim\limits_{x\to 3^{-}} f\left(x\right)$ 2	$\lim\limits_{x\to 3^{+}} f\left(x\right)$ 2	$\lim\limits_{x\to 3} f\left(x\right)$ 2 even though f(3) = n.p.
$\lim\limits_{x\to 4^{-}} f\left(x\right)$ 1	$\lim\limits_{x\to 4^{+}} f\left(x\right)$ 1	$\lim\limits_{x\to 4} f\left(x\right)$ 1 and f(4) = 1

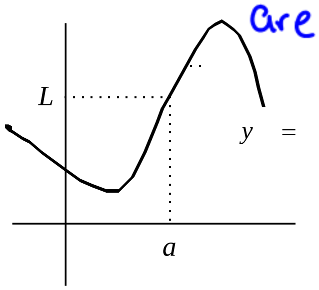
$$\lim_{x\to a} f\left(x\right)=L \quad \Leftrightarrow \quad \lim_{x\to a^{-}} f\left(x\right)=\lim_{x\to a^{+}} f\left(x\right)=L$$

What is the relationship between $f(a)$ and $\lim_{x \to a} f(x)$?

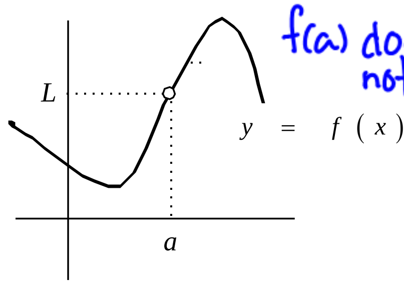
$\lim_{x \to a}$ and $f(a)$ are equal

$\lim_{x \to a}$ exists, but $f(a)$ does not

$\lim_{x \to a}$ and $f(a)$ are not the same

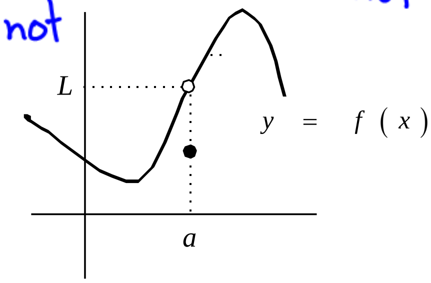


$f(a) = \lim_{x \to a} f(x) = L$



$\lim_{x \to a} f(x) = L$

but $f(a)$ is undefined



$\lim_{x \to a} f(x) = L$

and $f(a) \neq L$

limit suggests what $f(a)$ might be, but it might not be correct.

Properties of Limits

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

1) $\lim_{x \to c} \left(f(x) + g(x) \right) =$ $\lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$

The limit of the sum of two functions is the sum of their limits.

2) $\lim_{x \to c} \left(f(x) \bullet g(x) \right) =$ $\lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = L \cdot M$

The limit of a product of two functions is the product of their limits.

3) $\lim_{x \to c} k \bullet f(x) =$ $k \cdot L$ eg $\lim_{x \to 2} 3x^2 = 3 \cdot \lim_{x \to 2} x^2$

The limit of a constant times a function is the constant times the limit of the function.

4) $\lim_{x \to c} \dfrac{f(x)}{g(x)} =$ $\dfrac{L}{M}$

The limit of the quotient of two function is the quotient of their limits.

5) $\lim_{x \to c} \left(f(x) \right)^{\frac{r}{t}} =$ $[L]^{\frac{r}{t}}$

The limit of a rational power of a function is that power of the limit of the function.

Determine the following limits:

1. $\lim_{x\to 5} 3$ 3	2. $\lim_{x\to c} 3$ 3	3. $\lim_{x\to c} k$ k
4. $\lim_{x\to 3} x$ 3	5. $\lim_{x\to -5} x$ -5	6. $\lim_{x\to a} x$ a

Using the properties of limits, we can see that

$$\lim_{x\to 3} 5x^2 - 7x + 4 = \lim_{x\to 3} 5x^2 - \lim_{x\to 3} 7x + \lim_{x\to 3} 4$$

$$= 5\lim_{x\to 3} x^2 - 7\lim_{x\to 3} x + \lim_{x\to 3} 4$$

$$= 5\left(\lim_{x\to 3} x\right)^2 - 7\lim_{x\to 3} x + \lim_{x\to 3} 4$$

Two further properties of limits:

6) If $f(x)$ is a polynomial function, then $\lim_{x\to c} f(x) = f(c)$

The limit of a **polynomial** function can be found by direct substitution.

7) If $f(x)$ and $g(x)$ are polynomial functions, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$

The limit of a **rational function** can be found by direct substitution provided the denominator doesn't equal zero.

Determine the following limits:

1) $\lim_{x\to 3} 2x+5$

$= 2(3)+5$

$= 11$

2) $\lim_{x\to 5} \sqrt{4x^2+x}$

$= \sqrt{4(5)^2+5}$

$= \sqrt{105}$

3) $\lim_{x\to 2} \frac{x+2}{x^2+3x+1}$

$= \frac{2+2}{4+6+1}$

$= \frac{4}{11}$

4) $\lim_{x\to 0} \cos x$

$= \cos(0)$

$= 1$

How do you evaluate limits when you can't use direct substitution?

Strategy – **Factor** and cancel common factors.

5) $\lim_{x\to 2} \frac{x-2}{x^2-4}$

$= \lim_{x\to 2} \frac{\cancel{x-2}}{(x+2)\cancel{(x-2)}}$

$= \frac{1}{(2)+2}$

$= \frac{1}{4}$

6) $\lim_{x\to -4} \frac{x+4}{x^2+6x+8}$

$= \lim_{x\to -4} \frac{\cancel{x+4}}{(x+2)\cancel{(x+4)}}$

$= \frac{1}{-4+2}$

$= -\frac{1}{2}$

Factoring note: Sums and differences of cubes can be factored according to the following:

$$x^3 - y^3 = (x-y)(x^2+xy+y^2) \qquad x^3 + y^3 = (x+y)(x^2-xy+y^2)$$

Note that these are both the product of a binomial and a trinomial.

Thus $x^3 - 27 = (x-3)(x^2+3x+9)$

$27x^3 + 8y^3 = (3x+2y)(9x^2-6xy+4y^2)$

7) $\lim_{x\to -3} \frac{x+3}{2x^3+54}$

$\lim_{x\to -3} \frac{x+3}{2(x^3+27)}$

$\lim_{x\to -3} \frac{\cancel{x+3}}{2\cancel{(x+3)}(x^2-3x+9)}$

$= \frac{1}{2(9+9+9)}$

$= \frac{1}{54}$

8) $\lim_{x\to 4} \frac{\frac{1}{x}-\frac{1}{4}}{3x-12} = \frac{\frac{4}{4x}-\frac{x}{4x}}{3(x-4)}$

$= \frac{\frac{4-x}{4x}}{3(x-4)}$

$= \frac{-1\cancel{(x-4)}}{4x} \cdot \frac{1}{3\cancel{(x-4)}}$

$= \frac{-1}{48}$

Strategy – If limit involves square roots, multiply by the **conjugate**.

9) $\lim\limits_{x\to 9}\dfrac{x-9}{\sqrt{x}-3}\cdot\dfrac{\sqrt{x}+3}{\sqrt{x}+3}$

$$\lim_{x\to 9}\frac{\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{x-9}}$$

$$=\sqrt{9}+3$$

$$=6$$

10) $\lim\limits_{x\to 16}\dfrac{4-\sqrt{x}}{16-x}\cdot\dfrac{4+\sqrt{x}}{4+\sqrt{x}}$

$$=\lim_{x\to 16}\frac{\cancel{16-x}}{\cancel{(16-x)}(4+\sqrt{x})}$$

$$=\frac{1}{4+\sqrt{16}}$$

$$=\frac{1}{8}$$

$$(a-b)(a+b)=a^2-b^2$$

11) $\lim\limits_{x\to 3}\dfrac{\sqrt{x+6}-x}{3-x}\cdot\dfrac{\sqrt{x+6}+x}{\sqrt{x+6}+x}$

$$\lim_{x\to 3}\frac{x+6-x^2}{(3-x)(\sqrt{x+6}+x)}$$

$$\lim_{x\to 3}\frac{(\cancel{-1})(x^2-x-6)}{(\cancel{-1})(x-3)(\sqrt{x+6}+x)}$$

$$\lim_{x\to 3}\frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(\sqrt{x+6}+x)}$$

$$=\frac{5}{\sqrt{9}+3}=\frac{5}{6}$$

12) $\lim\limits_{x\to 6}\dfrac{6-x}{\sqrt{2x+4}-4}\cdot\dfrac{\sqrt{2x+4}+4}{\sqrt{2x+4}+4}$

$$\lim_{x\to 6}\frac{-(x-6)(\sqrt{2x+4}+4)}{2x+4-16}$$

$$\lim_{x\to 6}\frac{-(x-6)(\sqrt{2x+4}+4)}{2x-12}$$

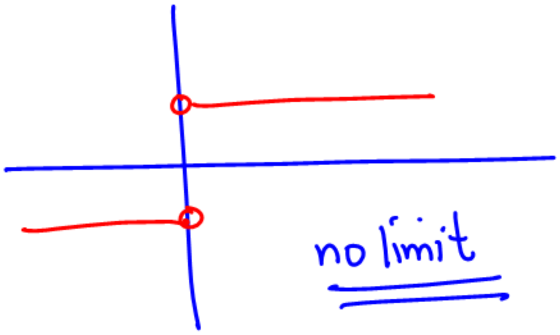
$$\lim_{x\to 6}\frac{-\cancel{(x-6)}(\sqrt{2x+4}+4)}{2\cancel{(x-6)}}$$

$$\frac{-(4+4)}{2}$$

$$=-4$$

Strategy – **Graph** the function and use the graph to help analyze the function or use a **numerical** approach.

9) $\lim_{x\to 0} \frac{x}{|x|}$ $\qquad \frac{f(x)}{|f(x)|}$



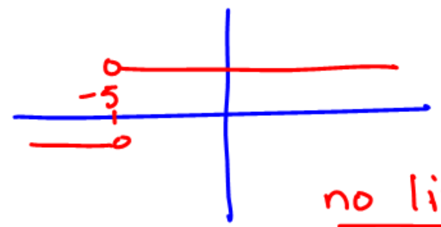
no limit

$f(.1) = 1$

$f(-.1) = -1$

} if the limit exists, these should be very close together.

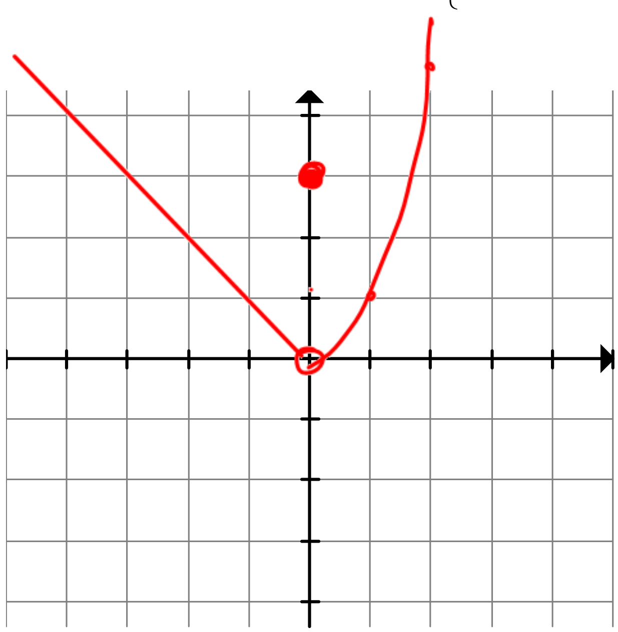
10) $\lim_{x\to -5} \frac{x+5}{|x+5|}$ $\qquad \frac{f(x)}{|f(x)|}$



no limit

that they are not close indicates that there is no limit

11) Given the function $f(x) = \begin{cases} -x & x<0 \\ 3 & x=0 \\ x^2 & x>0 \end{cases}$, determine the following



$\lim_{x\to -4} f(x) = -x$
$= -(-4)$
$= 4$

$\lim_{x\to 0^-} f(x) = -(x)$
$= -(0)$
$= 0$

$\lim_{x\to 0^+} f(x) = 0$
$= (x)^2$
$= (0)^2$

$\lim_{x\to 0} f(x) = 0$

$\lim_{x\to 3} f(x) = x^2$
$= (3)^2$
$= 9$

HW p 62 # 1-25, 31-43 odds.