11.2 Combinations

Investigate

1. If 5 sprinters compete in a race, how many different ways can the medals for first, second and third place, be awarded?

   ${}_5P_3$

   Does the order of finish for the fastest three matter here? order does matter

   This is an example of a **permutation** of __5__ objects taken __3__ at a time.

2. If 5 sprinters compete in a race and the fastest 3 qualify for the relay team, how many different relay teams can be formed?

   Visualize the 5 sprinters below. Since 3 will qualify for the relay team and 2 will not, consider the number of ways of arranging 3 Y's and 2 N's.

1	2	3		
Y	Y	Y	N	N
2	1	3		
3	1	2		

   Does the order of finish for the fastest three matter here? no

   This is an example of a **combination** of __5__ objects taken __3__ at a time.

Combinations

- An *unordered* arrangement of distinct objects is called a *combination*.
- The number of combinations of *n* distinct objects taken *r* at a time is

$${}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} \quad \text{with} \quad n \geq r \geq 0 \ .$$

Permutations

$${}_nP_r = \frac{n!}{(n-r)!}$$

Combination

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

"n choose r"

$${}_{49}C_7 = \frac{49!}{7!\,42!} = 85{,}900{,}584$$

Example 1:

a) How many different committees of 3 people can be formed from 7 people?

order does not matter $\quad {}_7C_3 = \frac{7!}{3!\,4!}$

b) How many different committees of 3 people can be formed from 7 people if the first person selected serves as the chairperson, the second as the treasurer, and the third as the secretary?

order does matter $\quad {}_7P_3 = \frac{7!}{4!}$

c) If the group of 7 people consists of 3 males and 4 females, how many different committees of 3 people can be formed with 1 male and 2 females?

Think: you must choose 1 male out of the group of 3 males and 2 females out of the group of 4 females.

1 male $\quad$ <u>and</u> $\quad$ 2 females $\quad \frac{4!}{2!\,2!} = \frac{24}{4}$

$${}_3C_1 \times {}_4C_2$$

$$3 \times 6 = \boxed{18}$$

d) If the group of 7 people consists of 3 males and 4 females, how many different committees of 3 people can be formed with at least one male on the committee?

<u>1m and 2f</u> or 2m 1f or 3m 0f

$${}_3C_1 \times {}_4C_2 + {}_3C_2 \times {}_4C_1 + {}_3C_3 \times {}_4C_0$$

$$\frac{3!}{2!\,1!} \times \frac{4!}{2!\,2!} + \frac{3!}{1!\,2!} \times \frac{4!}{3!\,1!} + \frac{3!}{3!\,0!} \times \frac{4!}{4!\,0!}$$

$$18 + 12 + 1 = \boxed{31}$$

Example 2:

a) Evaluate: $\frac{100!}{3!97!} =$

$${}_{100}C_3$$

or $\frac{100 \times 99 \times 98 \times \cancel{97!}}{3! \cdot \cancel{97!}} = 161700$

b) Solve: ${}_nC_2 = 21$

$$\frac{n!}{2!(n-2)!} = 21$$

$$\frac{n \times (n-1) \times \cancel{(n-2)!}}{2\,\cancel{(n-2)!}} = 21$$

$$n^2 - n = 42$$

$$n^2 - n - 42 = 0$$

$$(n-7)(n+6) = 0$$

$$\boxed{n = 7}$$

Example 3:
A standard deck of 52 playing cards consists of 4 suits (spaces, hearts, diamonds, and clubs) of 13 cards each.

	Club	Diamond	Heart	Spade
K	♣	♦	♥	♠
Q	♣	♦	♥	♠
J	♣	♦	♥	♠
10	♣	♦	♥	♠
9	♣	♦	♥	♠
8	♣	♦	♥	♠
7	♣	♦	♥	♠
6	♣	♦	♥	♠
5	♣	♦	♥	♠
4	♣	♦	♥	♠
3	♣	♦	♥	♠
2	♣	♦	♥	♠
A	♣	♦	♥	♠

a) How many different 5-card hands can be formed?

$_{52}C_5$

b) How many different 5-card hands can be formed that consist of all hearts?

$_{13}C_5$

c) How many different 5-card hands can be formed that consist of all face cards? K, Q, J

$_{12}C_5$

d) How many different 5-card hands can be formed that consist of 3 hearts and 2 spades?

$_{13}C_3 \times {}_{13}C_2$

e) How many different 5-card hands can be formed that consist of exactly 3 hearts? and 2 not hearts

$_{13}C_3 \times {}_{39}C_2$

f) How many different 5-card hands can be formed that consist of at least 3 hearts?

3H and 2non or 4H and 1 non or 5H and 0 non

$_{13}C_3 \times {}_{39}C_2 + {}_{13}C_4 \times {}_{39}C_1 + {}_{13}C_5 \times {}_{39}C_0$

$286 \times 741 + 715 \times 39 + 1287 \times 1$

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Example 4
A basketball coach has five guards and seven forwards on his basketball team.

a) In how many different ways can he select a starting team of two guards and three forwards?

$${}_5C_2 \times {}_7C_3$$

b) How many starting teams are there if the star player, who plays guard, must be included?

1 star and 1 other guard and 3 forwards

$${}_1C_1 \times {}_4C_1 \times {}_7C_3$$

Example 5

a) Explain in words why ${}_{10}C_7$ is the same as ${}_{10}C_3$

choosing which 7 people to give candy to is the same as choosing which 3 people do not receive

b) Use factorial notation to show ${}_{10}C_7 = {}_{10}C_3$

$$\frac{10!}{7!\cdot 3!} = \frac{10!}{3!\cdot 7!}$$

c) Prove the identity: ${}_nC_r = {}_nC_{n-r}$

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!\,(n-[n-r])!}$$

$$= \frac{n!}{(n-r)!\,(r)!}$$

Example 6
During a basketball tournament, all players shake hands with each other at the end of the last game. If 300 handshakes were exchanged, how many players were at the tournament?

n = # of players.

$${}_nC_2 = 300$$

$$\frac{n!}{2!\,(n-2)!} = 300$$

$$\frac{(n)(n-1)\cancel{(n-2)!}}{2\,\cancel{(n-2)!}} = 300$$

$$n(n-1) = 600$$

$$n = 25.$$