8.3 The Laws of Logarithms

Investigating Laws of Logarithms

Use your calculator to calculate the values in each of the tables below. Then compare the answers in the two columns and suggest a possible law

$(\log 100)(\log 10)$	2	$\log 1000$	3
$\log 12 + \log 2$	1.38	$\log 30$	1.48
$\log 6 + \log 5$	1.48	$\log 56$	1.75
$\log 8 + \log 7$	1.75	$\log 24$	1.38
Possible Law: $\log M + \log N =$ $\log(M \cdot N)$			

$\dfrac{\log 1000}{\log 200}$	1.30	$\log\left(\dfrac{1000}{200}\right)$	0.69
$\log 1000 - \log 200$	0.69	$\log 6$	0.77
$\log 40 - \log 200$	−0.69	$\log 0.2$	−0.69
$\log 30 - \log 5$	0.77	$\log 5$	0.69
Possible Law: $\log M - \log N =$ $\log\left(\frac{M}{N}\right)$			

$(\log 5)^2$	0.49	$\log 5^2$	1.4
$3\log 5$	2.1	$2\log 7$	1.7
$\log 49$	1.7 $49 = 7^2$	$\log 10\,000$	4
$4\log 10$	4	$\log 125$	2.1
Possible Law: $\log M^p =$ $p\log M$			

Product Law of Logarithms:	***Quotient Law of Logarithms:***
$\log_B(MN) = \log_B M + \log_B N$	$\log_B\left(\frac{M}{N}\right) = \log_B M - \log_B N$
Proof : Let M *and* N be any 2 positive numbers, and let $\log_B M = x$ and $\log_B N = y$. Also assume $B > 0$ *and* $B \neq 1$. $B^x = M$ $B^y = N$ $MN = M \cdot N$ $= B^x \cdot B^y$ $MN = B^{x+y}$ $\log_B(MN) = x+y$ $\log_B(MN) = \log_B M + \log_B N$	Proof : Let M *and* N be any 2 positive numbers, and let $\log_B M = x$ and $\log_B N = y$. Also assume $B > 0$ *and* $B \neq 1$. $B^x = M$ $B^y = N$ $\frac{M}{N} = \frac{M}{N}$ $= \frac{B^x}{B^y}$ $\frac{M}{N} = B^{x-y}$ $\log_B\left(\frac{M}{N}\right) = x - y = \log_B M - \log_B N$ eg: $\log 10 - \log 2$ $= \log\left(\frac{10}{2}\right)$ $= \log 5$

$\log_2 5 + \log_2 4 = \log_2 20$ $\log_4 10 = \log_4 5 + \log_4 2$

Power Law of Logarithms:	***Law of Logarithms for Roots***
$\log_B M^p = p\log_B M$	$\log_B \sqrt[p]{M} = \frac{1}{p}\log_B M$
Proof : Let M be any positive number, and let $\log_B M = x$. Also assume $B > 0$ *and* $B \neq 1$. $B^x = M$ $M^p = M^p$ $M^p = (B^x)^p$ $\log_B M^p = p \cdot x$ $\log_B M^p = p\log_B M$ $\log x^2 = 2\log x$ $\frac{1}{3}\log 5 = \log 5^{\frac{1}{3}}$ $= \log\sqrt[3]{5}$	Proof : This is just a natural extension of the Power Law of Logarithms because $\sqrt[p]{M} = M^{\frac{1}{p}}$ $\log\sqrt[p]{M} = \log M^{\frac{1}{p}}$ $= \frac{1}{p}\log M$ $\frac{1}{4}\log_5 x = \log_5\sqrt[4]{x}$ $\log_4\sqrt[3]{x^2} = \frac{2}{3}\log_4 x$

rational exponents are roots.

Examples

1) Rewrite each of the following in terms of the single logarithms of *x*, *y* and *z*

a) $\log_3 \frac{xy}{z^2}$ $= \log_3 x + \log_3 y - \log_3 z^2$

$= \boxed{\log_3 x + \log_3 y - 2\log_3 z}$

b) $\log_5 \sqrt{xy^4}$

$= \log_5 (xy^4)^{\frac{1}{2}} = \log_5 (x^{\frac{1}{2}} \cdot y^2) = \log_5 x^{\frac{1}{2}} + \log_5 y^2$

$= \boxed{\frac{1}{2}\log_5 x + 2\log_5 y}$

c) $\log_6 \frac{36}{y^3 z^2}$ $= \log_6 36 - \log_6 y^3 - \log_6 z^2$

$= \boxed{2 - 3\log_6 y - 2\log_6 z}$

2) Evaluate the following without calculators:

a) $\log_3 9\sqrt{27}$ $= \log_3 9 + \log_3 \sqrt{27}$ $\leftarrow \log_3 3^{\frac{3}{2}}$

$= \log_3 3^2 + \log_3 \sqrt{3^3}$

$= 2 + \frac{3}{2} = \boxed{3.5}$

b) $\log_4 48 + \log_4 \frac{2}{3} + \log_4 8$

$\log_4 \left(48 \times \frac{2}{3} \times 8\right) = \log_4 (256) = \log_4 4^4$

$= \boxed{4}$

c) $2\log_5 10 - \left(\log_5 50 + 3\log_5 \sqrt[3]{10}\right)$

$\log_5 10^2 - \log_5 50 - \log_5 \sqrt[3]{10}^3 = \log_5 \left(\frac{10^2}{50 \cdot 10}\right)$

$= \log_5 \left(\frac{100}{500}\right) = \log_5 \left(\frac{1}{5}\right)$

$= \boxed{-1}$

3) Express the following as a single logarithm:

a) $\log x + 3\log y - \frac{1}{2}\log w$ $= \log x + \log y^3 - \log w^{\frac{1}{2}}$

$= \log \left(\frac{x y^3}{w^{1/2}}\right)$

b) $2\log_3 x + 5\log_3 x - \dfrac{3\log_3 x}{2}$

$2\log_3 x + 5\log_3 x - \frac{3}{2}\log_3 x$ $\rightarrow \log_3 x^2 + \log_3 x^5 - \log_3 x^{\frac{3}{2}}$

$\log_3 \left(\frac{x^2 \cdot x^5}{x^{1.5}}\right) = \log_3 x^{5.5}$

$= \boxed{\log_3 x^{11/2}}$

c) $\log_7 (2x+2) - \log_7 \left(x^2 + 3x + 2\right)$

$\log_7 \left(\frac{2x+2}{x^2+3x+2}\right) = \log_7 \left(\frac{2\cancel{(x+1)}}{\cancel{(x+1)}(x+2)}\right) = \log_7 \left(\frac{2}{x+2}\right)$

4) What is the difference between the questions below?

a) Express 12 as a power of 3. b) $3^x = 12$ c) $\log_3 12 = x$

They are all asking "3 to what power makes 12"

5) Determine:

a) $\log_5 47 = x$

$5^x = 47$

$\log 5^x = \log 47$

$x \log 5 = \log 47$

$x = \frac{\log 47}{\log 5} = \log_5 47$

b) $9^x = 2$

$\log_9 2 = x$

$\frac{\log 2}{\log 9} = x$

*** Change of Base Formula**

$$\log_B A = \frac{\log A}{\log B} = \frac{\log_P A}{\log_P B} = \frac{\ln A}{\ln B}$$

6) The intensity level, β in decibels (dB), of a sound is defined to be $\beta = 10\log\frac{I}{I_0}$ where I is the intensity of the sound. The sound level of a chainsaw is about 85 dB, while that of a hairdryer is about 70 dB. How many times intense is the sound of a chainsaw compared to the sound of a hairdryer?

whisper 20dB] 10 x louder
library 30dB] 10x louder.
conversation 40dB

Δ dB = 85-70
= 15 dB

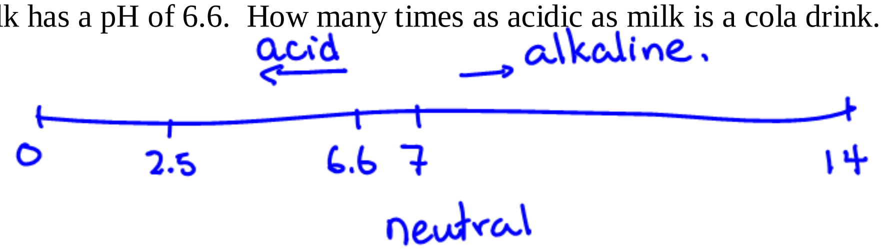
$\frac{15}{10} = \frac{10 \log I}{10}$

1.5 = log I

$10^{1.5} = I$
$= 31.6$

$\boxed{\text{a chainsaw is 31.6x louder than a hairdryer}}$

7) The pH scale measures the acidity or alkalinity of a solution, and is defined as $\text{pH} = -\log\left[\text{H}^+\right]$ where $\left[\text{H}^+\right]$ is the hydrogen ion concentration. A neutral solution has a pH of 7. A Cola drink has a pH of 2.5, while milk has a pH of 6.6. How many times as acidic as milk is a cola drink.



pH difference is 4.1

$10^{4.1} = 12589$

* cola is 12589x more acid than milk.

* milk is 12589x more alkaline than coke.