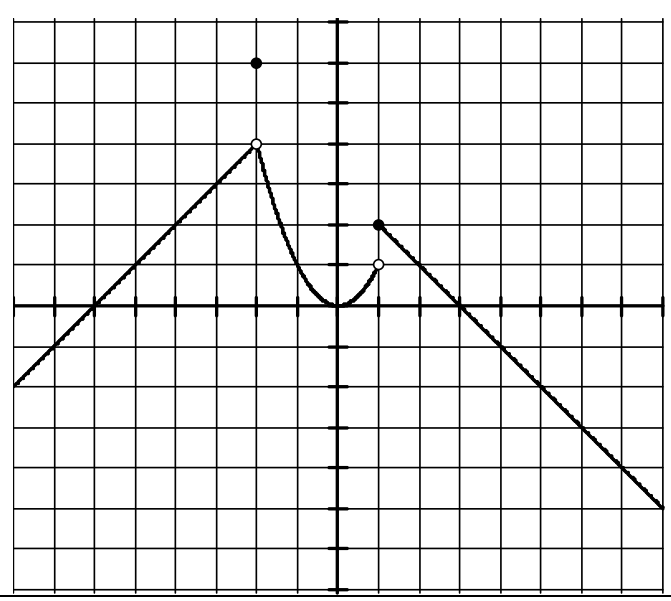
Warmup 1.3

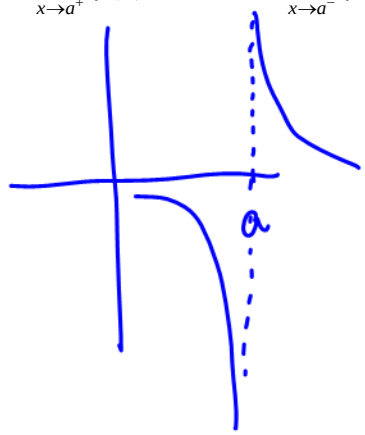
Determine the following limits

1. $\lim\limits_{x\to 5}\dfrac{2x+1}{3x-7}$ $= \dfrac{2(5)+1}{3(5)-7}$ $= \dfrac{11}{8}$	2. $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{x^2-9}$ $= \lim\limits_{x\to 3}\dfrac{\cancel{(x-3)}(x-2)}{(x+3)\cancel{(x-3)}}$ $= \dfrac{3-2}{3+3} = \dfrac{1}{6}$
3. $\lim\limits_{x\to 4}\dfrac{3+\sqrt{x}}{5-\sqrt{x-3}}$ $\lim\limits_{x\to 4}\dfrac{3+\sqrt{4}}{5-\sqrt{4-3}} = \dfrac{5}{4}$	4. $\lim\limits_{x\to 25}\dfrac{10-2\sqrt{x}}{3x-75}$ $\dfrac{2(5-\sqrt{x})(5+\sqrt{x})}{3(x-25)(5+\sqrt{x})}$ $\dfrac{2}{3}\,\dfrac{\cancel{25-x}\ -1}{(\cancel{x-25})(5+\sqrt{x})}$ $\dfrac{-2}{3(10)} = -\dfrac{1}{15}$
5. $\lim\limits_{x\to 0}\cos x$ $= \cos(0)$ $= 1$	6. $\lim\limits_{x\to \frac{\pi}{2}}\cos x$ $= \cos\left(\frac{\pi}{2}\right)$ $= 0$
7. $\lim\limits_{x\to 1}\dfrac{\lvert x-1\rvert}{x-1}$ graphically $\lim\limits_{x\to 1^+} = 1$, $\lim\limits_{x\to 1^-} = -1$ $\therefore$ no limit	8. $\lim\limits_{x\to e}(\ln x)$ $= \ln e$ $= 1$
9. Given the graph of $y = f(x)$ to the right, determine a) $\lim\limits_{x\to -2} f(x)$ $= 4$ b) $\lim\limits_{x\to -1} f(x)$ $= 1$ c) $\lim\limits_{x\to 1^+} f(x)$ 2 d) $\lim\limits_{x\to 1^-} f(x)$ 1 e) $\lim\limits_{x\to 1} f(x)$ no limit exists	

Limits Involving Infinity

***Vertical asymptotes*: Limits Equaling Infinity at finite values**

If $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$ then the graph appears as an asymptote.



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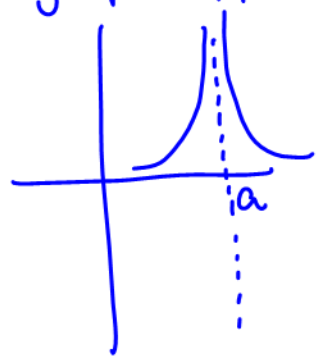


For a rational function of the form $f(x) = \frac{g(x)}{h(x)}$, $x = a$ is a vertical asymptote if $g(x) \neq 0$ and $h(x) = 0$

If both $g(a)$ and $h(a)$ equal zero, then there is either a hole in the graph or a break.

Find any vertical asymptotes. Describe the behavior to the left and right of any asymptotes.

1) $f(x) = \frac{1}{x^2}$

$x \neq 0$

try $x = .1$

$f(.1) = +$

$\therefore \lim_{x \to 0^+} = +\infty$

$x = -.1$

$f(-.1) = +$

$\therefore \lim_{x \to 0^-} = -\infty$

2) $f(x) = \frac{1}{x^2 - 4}$

$x \neq 2$

$f(2.1) = +$

$\therefore \lim_{x \to 2^+} f(x) = +\infty$

$f(1.9) = -$

$\therefore \lim_{x \to 2^-} f(x) = -\infty$

3) $f(x) = \frac{x + 2}{x^2 - 4}$

$= \frac{\cancel{x+2}}{(\cancel{x+2})(x-2)} = \frac{1}{x-2}$

$x \neq 2$

$\therefore$ hole at $x = -2$

$f(2.1) = +$

$\therefore \lim_{x \to 2^+} f(x) = +\infty$

$f(1.9) = -$

$\therefore \lim_{x \to 2^-} f(x) = -\infty$

4) $f(x) = \frac{x^2 - 4}{x + 2} = \frac{(\cancel{x+2})(x-2)}{(\cancel{x+2})} = x - 2.$

$\therefore$ hole at $x = -2$.

since denom $\neq 0$ then no asymptotes

5) $f(x) = \dfrac{x^2 - 5x - 6}{x^2 - 8x + 12} = \dfrac{(x-6)(x+1)}{(x-6)(x+2)}$

$f(x)$ has a hole at $x=6$
asymptote at $x=-2$

$f(-1.9) = -$

$\therefore \lim_{x \to -2^+} f(x) = -\infty$

$f(-2.1) = +$

$\therefore \lim_{x \to -2^-} f(x) = +\infty$

verify by graphing

6) $f(x) = \tan x$

by graphing

$\lim_{x \to \frac{\pi}{2}^+} f(x) = -\infty$

$\lim_{x \to \frac{\pi}{2}^-} f(x) = +\infty$

note: can also put in any $x = \frac{\pi}{2} + n\pi$

Finite limits as $x \to \pm\infty$ (Limits at the extreme left and right of the function) (what we call "end behaviour"

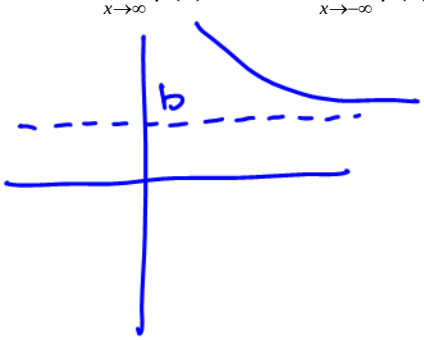
$$\lim_{x \to \infty} \frac{k}{x^n} = 0$$

$$\lim_{x \to -\infty} \frac{k}{x^n} = 0$$

← denominator becomes infinitely large

Horizontal asymptotes **– Limits at infinity**

If $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$ then there is a horizontal asymptote at $y=b$



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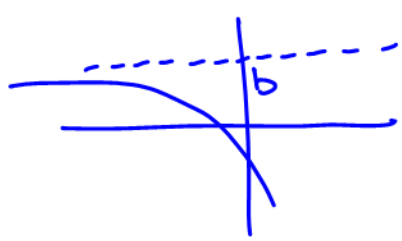

Note: The maximum number of horizontal asymptotes that a function can have is __2__. Not every function has a horizontal asymptote. A function may have a horizontal asymptote on one side, but not necessarily on the other side. A horizontal asymptote can be intersected by the function an infinite number of times.

the left and right side may have different horizontal asymptotes

Determine any horizontal asymptotes. To do this, it means that you have to examine the behavior of the function at both positive and negative infinity.

1) $f(x) = \dfrac{5x^2 + 7x}{13x^2 - 12}$

as $x \to +\infty$ or $x \to -\infty$

$5x^2 > 7x$ & $13x^2 > -12$

$\lim_{x \to +\infty} f(x) = \dfrac{5\cancel{x^2}}{13\cancel{x^2}} = \dfrac{5}{13}$ and $\lim_{x \to -\infty} f(x) = \dfrac{5}{13}$

2) $y = \dfrac{x^3 + 3x - 1}{4x^3 - 7}$

as $x \to \pm\infty$ x^3 and $4x^3$ are important

$\lim_{x \to +\infty} f(x) = \dfrac{\cancel{x^3}}{4\cancel{x^3}} = \dfrac{1}{4}$

$\lim_{x \to -\infty} = \dfrac{1}{4}$

3) $f(x) = \dfrac{2x^4 + 5x}{4x^3 - 1}$

$\lim_{x \to +\infty} f(x) = \dfrac{2x^4}{4x^3} = \dfrac{1}{2}x$

$= +\infty$

$\lim_{x \to -\infty} f(x) = -\infty$

4) $f(x) = \sin x$

no limit : as $x \to \infty$ or $-\infty$

it continues to oscillate between 1 and -1

5) $f(x) = \dfrac{\cos x}{x}$

as $x \to +\infty$ $\cos x$ oscillates between 0 and 1

but $x \to \infty$

as $x \to -\infty$ $\cos x$ still oscillates,

but $x \to -\infty$

$\lim_{x \to \pm\infty} f(x) = \dfrac{\text{something}}{\pm\infty} = 0$

6) $f(x) = e^x$

graphically, exponential function!

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to +\infty} f(x) = \infty$

7) $f(x) = \dfrac{5\sin x}{x^2 + 8x}$

$\lim_{x \to +\infty} f(x) = 0$

same reason as #5

$\lim_{x \to -\infty} f(x) = 0$

8) $y = \ln x$

graphically

no lefthand end behaviour

$\lim_{x \to +\infty} f(x) = +\infty$

It is sometimes helpful when exploring limits at infinity to use an ***end behavior model***. An end behavior model is a simple, basic function which closely approximates the actual function at either infinity or negative infinity.

For the function *f*:

The function *g* is a **right end behavior model** if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$

The function *g* is a **left end behavior model** if $\lim_{x\to-\infty}\frac{f(x)}{g(x)}=1$

For the following functions, determine right and left end behavior models:

Function	***Left end behavior model*** which grows most to left	***Right end behavior model*** which grows most to the right
$y=-4x^5-7x^3+3x^2-8$	$y=-4x^5$	$y=-4x^5$
$y=3^x+x^2$	$y=x^2$	$y=3^x$
$y=2x^3-5^{-x}$ $=2x^3-\frac{5}{x}$	$y=2x^3$	$y=2x^3$
$y=6x+\log\lvert x\rvert$ compare the graphs	$y=6x$	$y=6x$
$y=\sin\left(\frac{1}{x}\right)$ as $x\to\infty$ this looks like $\sin(0)$	$y=0$	$y=0$
$y=\sqrt{9x^2+2053}$	$y=\sqrt{9x^2}$ or $\lvert 3x\rvert$	$y=\sqrt{9x^2}$ or $\lvert 3x\rvert$
$y=\frac{1}{x}+3$ at ∞, $\frac{1}{x}\to 0$	$y=3$	$y=3$
$y=\frac{4x^3-5x-2}{12-3x^2+7x}$	$y=\frac{4x^3}{-3x^2}$	$y=\frac{4x^3}{-3x^2}$
$y=\frac{-18x}{\sqrt{36x^2+10000x}}=\frac{-18x}{\lvert 6x\rvert}$	$=\frac{-18x}{\lvert 6x\rvert}$	$=\frac{-18x}{\lvert 6x\rvert}$

Determine all asymptotes for the function $f(x) = \dfrac{4x}{\sqrt{x^2 - 4}}$

$x^2 - 4 = 0$

$x = \pm 2$

vertical asymptotes at $x = 2, -2$

$y = 1$ (right)

$y = -1$ (left)

end behaviour

Left

$y = \dfrac{4x}{|x|}$

$y = -1$ because $x = \ominus$

Right

$y = \dfrac{4x}{|x|}$

$y = +1$ because $x = \oplus$

"Seeing" limits at infinity – To explore limits at infinity on a graphics calculator, the question is how far do you have to go to see what is happening at infinity. One way to explore this graphically is to examine the graph of $y = f\left(\dfrac{1}{x}\right)$ and examine the behavior of this transformed function around zero.

This means $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to 0^+} f\left(\dfrac{1}{x}\right)$ and $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to 0^-} f\left(\dfrac{1}{x}\right)$